

a. (1p)

The electric charges of the plates are

$$q_{1,2} = Q \pm \frac{\epsilon_0 S U}{d}$$

$$F = 0 \Rightarrow q = 0 \Rightarrow Q = \frac{\epsilon_0 S U}{d}. \text{ The equilibrium is unstable.}$$

b. (1p)

$$m \frac{\ddot{x}}{2} = \frac{q_1 q_2}{2 \epsilon_0 S} \Rightarrow \ddot{x} = \frac{\epsilon_0 S U^2}{m} \left(\frac{1}{d^2} - \frac{1}{x^2} \right)$$

c. (2p)

$$\dot{x} \frac{d\dot{x}}{dt} = \frac{\epsilon_0 S U^2}{m} \left(\frac{\dot{x}}{d^2} - \frac{\dot{x}}{x^2} \right) \Rightarrow \frac{1}{2} \frac{dv^2}{dt} = \frac{\epsilon_0 S U^2}{m} \left(\frac{1}{d^2} \frac{dx}{dt} + \frac{d}{dt} \left(\frac{1}{x} \right) \right) \Rightarrow \frac{1}{2} \frac{dv^2}{dx} - \frac{\epsilon_0 S U^2}{m} \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{\epsilon_0 S U^2}{m} \frac{1}{d^2}$$

$$\frac{m}{2 \epsilon_0 S U^2} d(v^2) = \left(\frac{1}{d^2} - \frac{1}{x^2} \right) dx \Rightarrow v^2 = \frac{2 \epsilon_0 S U^2}{m} \left(\frac{x}{d^2} + \frac{1}{x} \right) + C$$

$$x = d \Rightarrow v = 0 \Rightarrow C = -\frac{4 \epsilon_0 S U^2}{md} \Rightarrow v^2 = \frac{2 \epsilon_0 S U^2}{m} \left(\frac{\sqrt{x}}{d} - \frac{1}{\sqrt{x}} \right)^2 = \frac{2 \epsilon_0 S U^2}{md^2} \frac{(x-d)^2}{x}$$

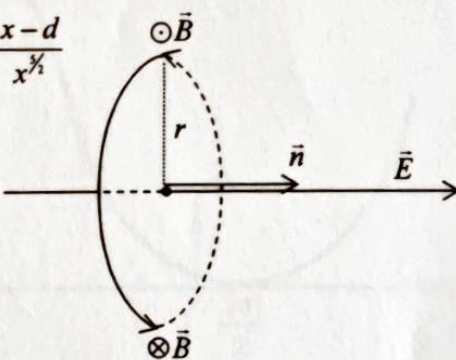
d. (1p)

$$E = \frac{U}{x} \Rightarrow \dot{E} = -\frac{U}{x^2} \dot{x} = -\frac{U^2}{d} \sqrt{\frac{2 \epsilon_0 S}{m}} \frac{x-d}{x^{3/2}}$$

e. (1p)

For an arbitrarily chosen orientation of the magnetic lines,

$$B \cdot 2\pi r = \frac{1}{c^2} \dot{\Phi}_{el} = \frac{1}{c^2} \pi r^2 \dot{E} \Rightarrow B(x) = -\frac{r U^2}{2c^2 d} \sqrt{\frac{2 \epsilon_0 S}{m}} \frac{x-d}{x^{3/2}}$$



f. (3p)

The e.m.f. induced in the ring by the magnetic field is

$$e(x) = -\dot{\Phi}_{mag} = -A \dot{B}.$$

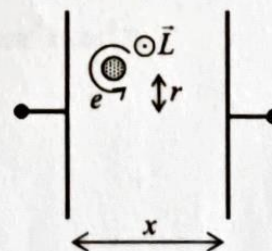
Let ρ be the radius of the ring. The torque exerted upon the ring is

$$M(x) = F \rho = q \frac{e}{2\pi \rho} \rho = -\frac{qA}{2\pi} \dot{B}.$$

$$\dot{L} = I \dot{\omega} = -\frac{qA}{2\pi} \dot{B} \Rightarrow \omega(x) = -\frac{qAB}{2\pi I} + C.$$

$$x = d \Rightarrow B = 0 \Rightarrow C = 0 \Rightarrow \omega(x) = -\frac{qAB}{2\pi I} = \frac{qArU^2}{4\pi Ic^2 d} \sqrt{\frac{2 \epsilon_0 S}{m}} \frac{x-d}{x^{3/2}}$$

$$\frac{d\omega}{dx} = 0 \Rightarrow \frac{x^{3/2} - \frac{3}{2}x^{1/2}(x-d)}{x^{10}} = 0 \Rightarrow \frac{5}{2}d = \frac{3}{2}x \Rightarrow x = \frac{5}{3}d \Rightarrow \omega_{max} = \frac{3qArU^2}{25\pi Ic^2 d^2} \sqrt{\frac{3 \epsilon_0 S}{10md}}$$



g. (1p)

$$x > d \Rightarrow E = \text{const.} \Rightarrow B = 0 \Rightarrow \omega = 0$$

1.

The mass of each of the 20 smaller cubes is $m_1 = m/20$. The length of the edge of each smaller cube is $l_1 = l/3$. So the moment of inertia of each smaller cube with respect to its own axis is $I_1 = I/180$. (1p)

There are 8 cubes whose axes are at a distance $l/3$ from the central axis. There are 12 cubes whose axes are at a distance $l\sqrt{2}/3$ from the central axis. According to the parallel axes theorem (Steiner)

$$I = 8 \left(I_1 + \frac{m}{20} \frac{l^2}{9} \right) + 12 \left(I_1 + \frac{m}{20} \frac{2l^2}{9} \right) \Rightarrow I = 20 \frac{I}{20 \cdot 9} + \frac{m}{20} \frac{l^2}{9} (8 + 24) \Rightarrow I = \frac{ml^2}{5} \quad (3p)$$

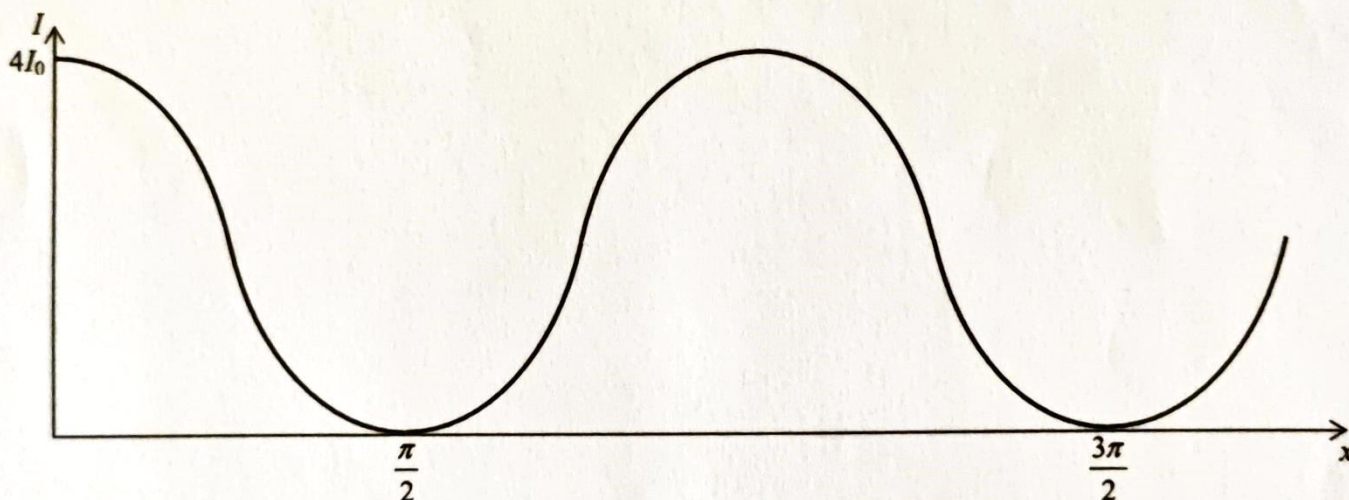
2.

a. (1p)

$$\delta = \frac{2l}{3^N} \sin \alpha$$

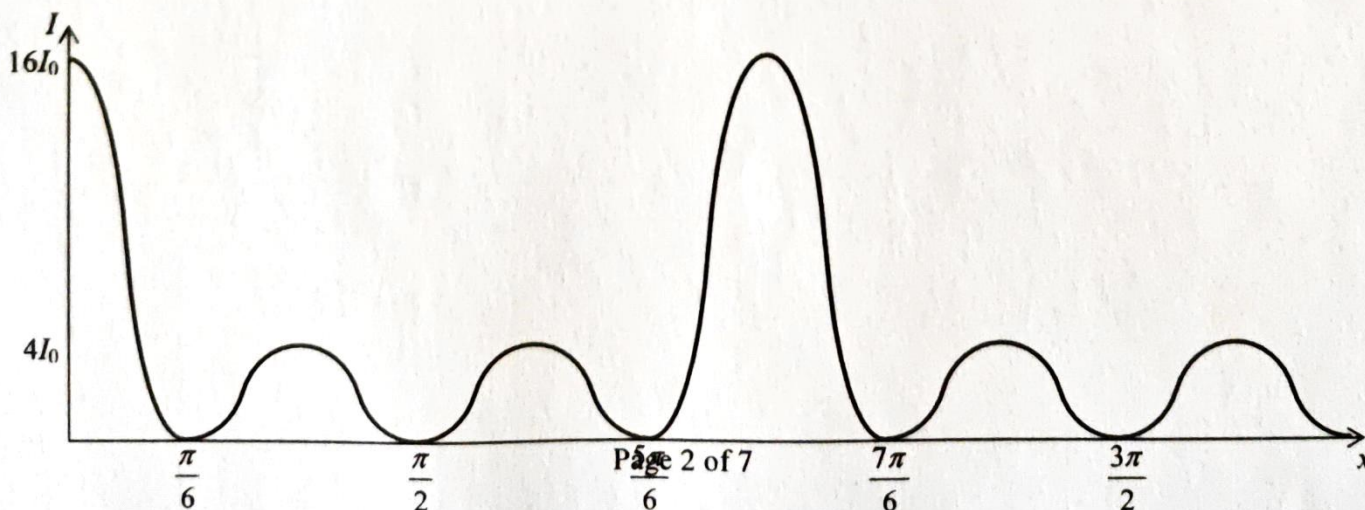
$$I(\alpha) = 2I_0 \left(1 + \cos \left(\frac{2\pi}{\lambda} \frac{2l}{3^N} \sin \alpha \right) \right) = 4I_0 \cos^2 \left(\frac{2\pi l}{3^N \lambda} \sin \alpha \right)$$

b. (1p)



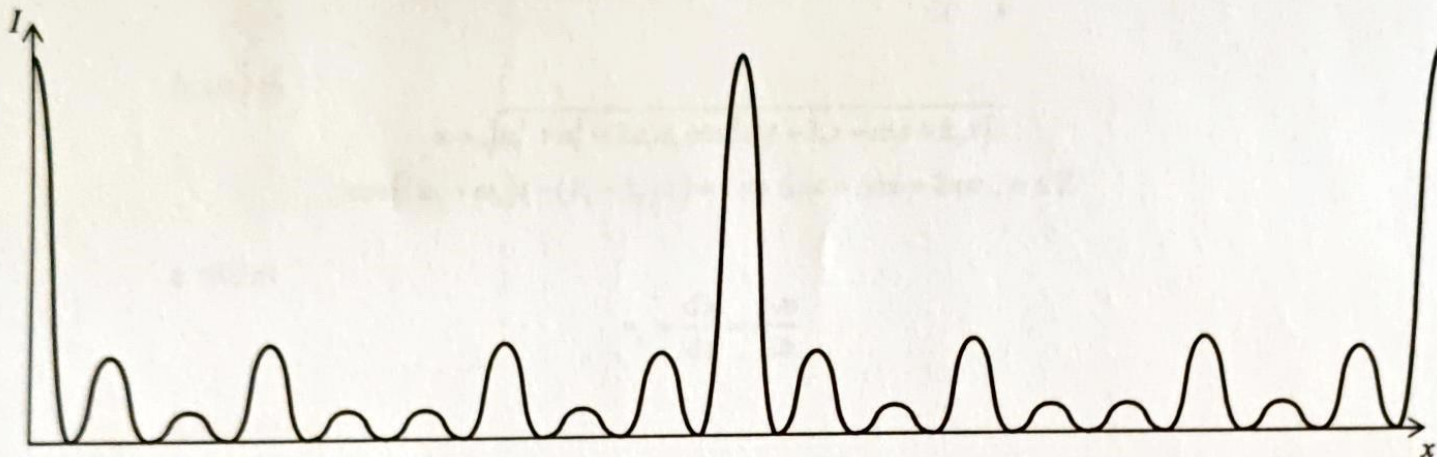
c. (1p) + (1p)

$$I(\alpha) = 8I_0 \cos^2 \left(\frac{2\pi l}{3^N \lambda} \sin \alpha \right) \left(1 + \cos \left(\frac{2\pi}{\lambda} \frac{2l}{3^{N-1}} \sin \alpha \right) \right) = 16I_0 \cos^2 x \cos^2 3x$$



d. (1p) + (1p)

$$I(\alpha) = 4^N I_0 \prod_{k=1}^N \cos^2 \left(\frac{2\pi l}{3^k \lambda} \sin \alpha \right) = 4^N I_0 \prod_{k=1}^N \cos^2 3^{k-1} x$$



So the diffraction pattern mimics perfectly the fractal character of the diffraction grating.

1.

a. (0.5p)

$$\omega t - kx = \text{const.} \Rightarrow \omega dt - k dx = 0 \Rightarrow c = \frac{dx}{dt} = \frac{\omega}{k}$$

b. (0.5p)

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos(\omega_1 t - k_1 x - \omega_2 t + k_2 x)}$$

$$\cos((\omega_1 - \omega_2)t - (k_1 - k_2)x) = 1 \Rightarrow \Delta\omega t - \Delta kx = 2n\pi, n \in \mathbb{Z}$$

c. (0.5p)

$$v_g = \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}$$

d. (1.5p)

$$\omega(k) = \omega_0 + \frac{\Delta\omega}{\Delta k}(k - k_0) \Rightarrow d\xi(x, t) = da \sin\left(\left(\omega_0 + \frac{\Delta\omega}{\Delta k}(k - k_0)\right)t - kx\right)$$

$$\xi(x, t) = \int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}} \frac{a}{\Delta k} \sin\left(\left(\omega_0 - \frac{\Delta\omega}{\Delta k}k_0\right)t + \left(\frac{\Delta\omega}{\Delta k}t - x\right)k\right) dk$$

$$\xi(x, t) = -\frac{a}{\Delta k} \frac{\cos\left(\left(\omega_0 - \frac{\Delta\omega}{\Delta k}k_0\right)t + \left(\frac{\Delta\omega}{\Delta k}t - x\right)k\right)}{\frac{\Delta\omega}{\Delta k}t - x} \Bigg|_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}}$$

$$\xi(x, t) = 2a \frac{\sin\left(\frac{\Delta\omega t - \Delta kx}{2}\right)}{\Delta\omega t - \Delta kx} \sin(\omega_0 t - k_0 x)$$

$$\frac{\sin\left(\frac{\Delta\omega t - \Delta kx}{2}\right)}{\Delta\omega t - \Delta kx} = \text{const.} \Rightarrow \Delta\omega t - \Delta kx = \text{const.} \Rightarrow v_g = \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}$$

Viewing the expression above as a function of x , the points on the x -axis characterized by maxima of the amplitude at some moment of time t satisfy the condition

$$f'(x) = 0 \Rightarrow \frac{-\Delta k \left[(\Delta\omega t - \Delta kx) \cos\left(\frac{\Delta\omega t - \Delta kx}{2}\right) \frac{1}{2} - \sin\left(\frac{\Delta\omega t - \Delta kx}{2}\right) \right]}{(\Delta\omega t - \Delta kx)^2} = 0$$

$$\tan\left(\frac{\Delta\omega t - \Delta kx}{2}\right) = \frac{\Delta\omega t - \Delta kx}{2} \Rightarrow \frac{\Delta\omega t - \Delta kx}{2} \approx \pm(2n+1)\frac{\pi}{2}, n \in \mathbb{N}^*$$

$$\Delta\omega t - \Delta kx = 0 \Rightarrow x = \frac{\Delta\omega t}{\Delta k} \Rightarrow f\left(\frac{\Delta\omega t}{\Delta k}\right) = 1$$

$$\Delta\omega t - \Delta kx = 3\pi \Rightarrow x = \frac{\Delta\omega t - 3\pi}{\Delta k} \Rightarrow f\left(\frac{\Delta\omega t - 3\pi}{\Delta k}\right) = \frac{\sin\left(\frac{3\pi}{2}\right)}{3\pi} = -\frac{1}{3\pi}$$

$$\left| \frac{f\left(\frac{\Delta\omega t - 3\pi}{\Delta k}\right)}{f\left(\frac{\Delta\omega t}{\Delta k}\right)} \right| = \frac{1}{3\pi} \approx 0.1$$

2.

e. (1p)

$$v_g = \frac{d\omega}{dk} = \frac{d\omega}{dE} \frac{dE}{dk} = \frac{d\omega}{dE} \frac{dE}{dp} \frac{dp}{dk} = \frac{1}{\hbar} v \hbar = v$$

f. (1p)

$$\sin\left(\frac{\Delta\omega t - \Delta kx}{2}\right) = 0 \Rightarrow \frac{\Delta\omega t - \Delta kx}{2} = n\pi, n \in \mathbb{N}^* \Rightarrow x = \frac{\Delta\omega t - 2\pi}{\Delta k}$$

$$\Delta x = \frac{\Delta\omega t - 2\pi}{\Delta k} - \frac{\Delta\omega t}{\Delta k} = \frac{2\pi}{\Delta k} = \frac{2\pi}{\frac{\Delta p}{\hbar}} \Rightarrow \Delta x \Delta p = \frac{h}{\pi} \pi = h$$

g. (1p)

$$\Psi(x, t) = a [\cos(\omega t + kx) + i \sin(\omega t + kx)] = a e^{i(\omega t + kx)}$$

$$\sin(\omega t + kx) = \frac{e^{i(\omega t + kx)} - e^{-i(\omega t + kx)}}{2i}$$

$$\cos(\omega t + kx) = \frac{e^{i(\omega t + kx)} + e^{-i(\omega t + kx)}}{2}$$

h. (0.5p)

$$\Psi(x, t) = a e^{i(\omega t - kx)} + a e^{i(\omega t + kx)} = a e^{i\omega t} 2 \frac{e^{-ikx} + e^{ikx}}{2} = 2a \cos kx e^{i\omega t}$$

i. (0.25p)

$$E = 0; |\Psi(x, t)|^2 = \frac{1}{l}$$

j. (0.5p)

$$l = n \frac{\lambda}{2}; n \in \mathbb{N}^* \Rightarrow \lambda_n = \frac{2l}{n} \Rightarrow p_n = \frac{h}{\lambda_n} = n \frac{\pi \hbar}{l} \Rightarrow E_n = \frac{p_n^2}{2m} = \frac{\pi^2 \hbar^2}{2ml^2} n^2$$

$$E_1 = \frac{\pi^2 \hbar^2}{2ml^2} = \frac{(\pi \hbar c)^2}{2E_0 l^2} = \frac{(3.14 \cdot 10^{-34} \text{ Js} \cdot 3 \cdot 10^8 \text{ m/s})^2}{2 \cdot 0.5 \text{ MeV} \cdot 10^{-20} \text{ m}^2} = 34.8 \text{ eV}$$

k. (0.5p)

$$\Psi_1(x, t) = c_1 \cos k_1 x e^{i\omega_1 t} = c_1 \cos\left(\frac{\pi}{l} x\right) e^{i\omega_1 t}$$

$$\Psi_2(x, t) = c_2 \sin k_2 x e^{i\omega_2 t} = c_2 \sin\left(\frac{2\pi}{l} x\right) e^{i\omega_2 t}$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} |\Psi_1(x, t)|^2 dx = 1 \Rightarrow c_1^2 \int_{-\frac{l}{2}}^{\frac{l}{2}} \cos^2\left(\frac{\pi}{l} x\right) dx = 1 \Rightarrow c_1 = \sqrt{\frac{2}{l}}$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} |\Psi_2(x, t)|^2 dx = 1 \Rightarrow c_2^2 \int_{-\frac{l}{2}}^{\frac{l}{2}} \sin^2\left(\frac{2\pi}{l} x\right) dx = 1 \Rightarrow c_2 = \sqrt{\frac{2}{l}}$$

$$P_1\left(\left[-\frac{l}{6}, +\frac{l}{6}\right]\right) = \int_{-\frac{l}{6}}^{\frac{l}{6}} |\Psi_1(x, t)|^2 dx = \frac{2}{l} \int_{-\frac{l}{6}}^{\frac{l}{6}} \frac{1 + \cos\left(\frac{2\pi x}{l}\right)}{2} dx = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}$$

$$P_2\left(\left[-\frac{l}{6}, +\frac{l}{6}\right]\right) = \int_{-\frac{l}{6}}^{\frac{l}{6}} |\Psi_2(x, t)|^2 dx = \frac{2}{l} \int_{-\frac{l}{6}}^{\frac{l}{6}} \frac{1 - \cos\left(\frac{4\pi x}{l}\right)}{2} dx = \frac{1}{3} - \frac{\sqrt{3}}{4\pi}$$

l. (0.5p)

$$\Psi(x, t) = \alpha_1 c_1 \cos k_1 x e^{i\omega_1 t} + \alpha_2 c_2 \sin k_2 x e^{i\omega_2 t}; \Psi^*(x, t) = \alpha_1 c_1 \cos k_1 x e^{-i\omega_1 t} + \alpha_2 c_2 \sin k_2 x e^{-i\omega_2 t}$$

$$|\Psi(x, t)|^2 = \alpha_1^2 c_1^2 \cos^2 k_1 x + \alpha_2^2 c_2^2 \sin^2 k_2 x + \alpha_1 \alpha_2 c_1 c_2 \cos k_1 x \sin k_2 x \left(e^{i(\omega_1 - \omega_2)t} + e^{i(\omega_2 - \omega_1)t} \right)$$

$$|\Psi(x, t)|^2 = \frac{2}{l} \left[\alpha_1^2 \cos^2 k_1 x + \alpha_2^2 \sin^2 k_2 x + 2\alpha_1 \alpha_2 \cos k_1 x \sin k_2 x \cos((\omega_1 - \omega_2)t) \right]$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} |\Psi(x, t)|^2 dx = 1 \Rightarrow \alpha_1^2 \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{1 + \cos 2k_1 x}{2} dx + \alpha_2^2 \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{1 - \cos 2k_2 x}{2} dx +$$

$$+ \alpha_1 \alpha_2 \cos((\omega_1 - \omega_2)t) \int_{-\frac{l}{2}}^{\frac{l}{2}} [\sin((k_2 + k_1)t) - \sin((k_2 - k_1)t)] dx = \frac{l}{2}$$

$$\alpha_1^2 + \alpha_2^2 = 1$$

m. (0.25p)

$$E = 0; |\Psi(x, y, z, t)|^2 = \frac{1}{l_x l_y l_z}$$

n. (0.25p)

The movement of the particle on each axis is independent, so

$$E_{n_x, n_y, n_z} = E_{n_x} + E_{n_y} + E_{n_z} = \frac{\pi^2 \hbar^2 c^2}{2E_0} \left(\frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2} + \frac{n_z^2}{l_z^2} \right)$$

o. (0.25p)

$E_{1,1,1} = 3E_1 \approx 104.43 \text{ eV}$. The ground state is not degenerate.

p. (0.5p)

$$l_2 = 2l_1 ; n_x = 5 \text{ and } n_y = 8 ; n_x' = 4 \text{ and } n_y' = 10$$

q. (0.5p)

$$\Psi_{n_x, n_y, n_z}(x, y, z, t) = c_{n_x, n_y, n_z} \sin\left(k_{n_x} x + n_x \frac{\pi}{2}\right) \sin\left(k_{n_y} y + n_y \frac{\pi}{2}\right) \sin\left(k_{n_z} z + n_z \frac{\pi}{2}\right) e^{i(\omega_{n_x} + \omega_{n_y} + \omega_{n_z})t}$$

$$\int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} \int_{-\frac{l_y}{2}}^{\frac{l_y}{2}} \int_{-\frac{l_z}{2}}^{\frac{l_z}{2}} |\Psi_{n_x, n_y, n_z}(x, y, z, t)|^2 dx dy dz = 1$$

$$c_{n_x, n_y, n_z}^2 \int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} \frac{1 \pm \cos(2k_{n_x} x)}{2} dx \int_{-\frac{l_y}{2}}^{\frac{l_y}{2}} \frac{1 \pm \cos(2k_{n_y} y)}{2} dy \int_{-\frac{l_z}{2}}^{\frac{l_z}{2}} \frac{1 \pm \cos(2k_{n_z} z)}{2} dz = 1$$

$$c_{n_x, n_y, n_z}^2 \frac{l_x}{2} \frac{l_y}{2} \frac{l_z}{2} = 1 \Rightarrow c_{n_x, n_y, n_z} = \sqrt{\frac{8}{l_x l_y l_z}}$$