



3rd Romanian Master of Sciences 2010

Physics – Theoretical Tour

A. ELECTRICITY

a) $I_F < 200 \text{ mA}$ (the fuse is intact):

The fuse F acts as a short-circuit, and the voltage across it vanishes. No electric current I_1 flows in R_1 , $I_1 = 0$.

0.25 p

No electric current flows in L_2 for $t = 0$, the voltage across the fuse is zero, the electric current I_2 is zero, $I_2 = 0$.

0.25 p

Across the inductance L_1 a constant voltage V causes I to increase at a constant rate $V/L_1 = 1000 \text{ A/s}$.

0.25 p

All electrical current I flows in the fuse, $I_F = I$.

0.25 p

The melting condition is realized for $t = L_1 I_F / V = 0.2 \text{ ms}$

0.5 p

total a) 1.5 p

b) Once the fuse melts, the current I_F vanishes, $I_F = 0$.

0.25 p

Right after the fuse melts, the electric current I conserves its value before melting, $I = 200 \text{ mA}$.

0.25 p

The current flowing in L_2 is free of jumps (discontinuities). Then, right after melting the fuse $I_2 = 0$.

0.25 p

As a consequence of Kirchhoff's first law, right after melting the fuse, the current I_1 flowing in R_1 is 200 mA, causing a voltage drop across R_1 of 200 V, with the "+" pole in the right hand side.

0.25 p

As a consequence, a voltage across L_1 develops $V - R_1 I_F = 200 \text{ V} - 10 \text{ V} = 190 \text{ V}$, causing the variation of I at a rate $\Delta I / \Delta t$

given by $(V - R_1 I_F) / L_1 = 19000 \text{ A/s}$. Right after the fuse melts, the voltage polarity causes I to decrease at the rate above.

0.5 p

The 200 V voltage drop across R_1 produces an increase of I_2 . Right after the fuse melts, the voltage drop across R_2 is zero, therefore, immediately after melting the fuse, I_2 raises at a rate of 40000 A/s .

0.25 p

From the first Kirchhoff law, it follows that immediately after the fuse melts, the current I_1 falls at a rate of 59000 A/s .

0.5 p

total b) 2.25 p

c) For t approaching infinity:

$I_1 = 10 \text{ mA}$

0.25 p

$I_2 = 50 \text{ mA}$

0.25 p

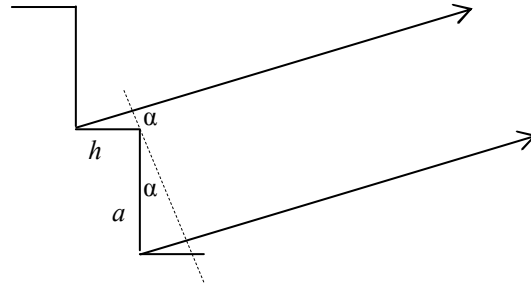
$I = 60 \text{ mA}$

0.25 p

total c) 0.75 p

B. MICHELSON'S "LADDER"

a. 1.5p



It is obvious that for $\alpha = 0$, the path length difference δ between two neighboring light rays is $(n - 1)h$. This difference increases with α .

$$\delta = nh + a \sin \alpha - h \cos \alpha = (n - 1)h + a \sin \alpha + h(1 - \cos \alpha) .$$

$$\delta = k\lambda , k \in \mathbb{N} .$$

One can see that in our specific example we get a diffraction maximum for $\alpha = 0$ and $k_0 = 10,000$. So the condition for the principal maxima can be written:

$$a \sin \alpha + h(1 - \cos \alpha) = p\lambda , p = k - k_0 > 0 .$$

b. 1.5p

One knows that the intensity of the light diffracted by a slit with aperture a depends on the deflecting angle according to:

$$I(\alpha) = I_0 \left[\frac{\sin \left(\frac{\pi a \sin \alpha}{\lambda} \right)}{\frac{\pi a \sin \alpha}{\lambda}} \right]^2 .$$

The first diffraction minimum occurs for

$$\frac{\pi a \sin \alpha}{\lambda} = \pi \Rightarrow \sin \alpha = \frac{\lambda}{a} = 5 \cdot 10^{-5} .$$

For such a small angle,

$$\delta \approx a\alpha + h \frac{\alpha^2}{2} \approx a\alpha \approx a \sin \alpha = p\lambda .$$

$$\sin \alpha < \frac{\lambda}{a} \Rightarrow p < 1 \Rightarrow p = 0 \Rightarrow k = k_0 .$$

So the only principal maximum that can be seen is the "central" one.

c. 1.5p

Increasing the wavelength of the light stream by $\Delta\lambda$ can lead to the overlapping of two such central maxima.

$$k_0\lambda = (k_0 - 1)(\lambda + \Delta\lambda) \Rightarrow \Delta\lambda \approx \frac{\lambda}{k_0} = 0.5\text{A} .$$



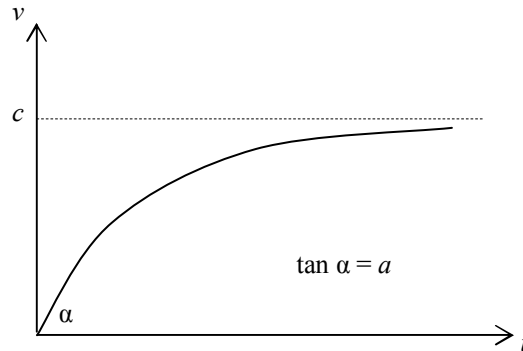
3rd Romanian Master of Sciences 2010

Physics – Theoretical Tour

SPECIAL RELATIVITY: ACCELERATING SPACESHIP

All quantities in the rocket's reference frame will be denoted with a prime (').

a. 1p



b. 2p

The momentum of the rocket increases uniformly with time.

$$\frac{dp_x}{dt} = \text{const} \Rightarrow \frac{v_x}{\sqrt{1 - \left(\frac{v_x}{c}\right)^2}} = at \Rightarrow v_x(t) = c \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}}.$$

(We took into account the fact that $v_x(0) = 0$.) The rocket's acceleration in the Earth's reference frame is

$$a_x(t) = \frac{dv_x}{dt} = \frac{a}{\left(1 + \left(\frac{at}{c}\right)^2\right)^{\frac{3}{2}}}.$$

The Lorentz transformation for the accelerations on the x -axis is

$$a_x = a'_x \left(\frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V}{c^2} v'_x} \right)^3.$$

For $V = v_x$, we get $v'_x = 0$, so the astronaut's "weight" will be given by

$$a'_x(t) = \frac{a_x}{\left(1 - \frac{v_x^2}{c^2}\right)^{\frac{3}{2}}} = a \Rightarrow W = ma.$$

c. 1p

$$v_x = \frac{dx}{dt} \Rightarrow dx = c \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} dt \Rightarrow x(t) = \frac{c^2}{a} \left[\sqrt{1 + \left(\frac{at}{c}\right)^2} - 1 \right].$$

d. 1p

The distance increases asymptotically with time:

$$\lim_{t \rightarrow \infty} \frac{x(t)}{t} = c ; \lim_{t \rightarrow \infty} (x(t) - ct) = -\frac{c^2}{a} .$$

The asymptote's equation is

$$x(t) = ct - \frac{c^2}{a} = c \left(t - \frac{c}{a} \right) .$$

For the drawing, see **g**.

e. 1p

As seen on the diagram, in the long run the spaceship's motion gets infinitely close to the motion of a light signal emitted from Earth at Earth time c/a .

f. 1.5p

The Earth time when the first signal reaches the rocket is given by

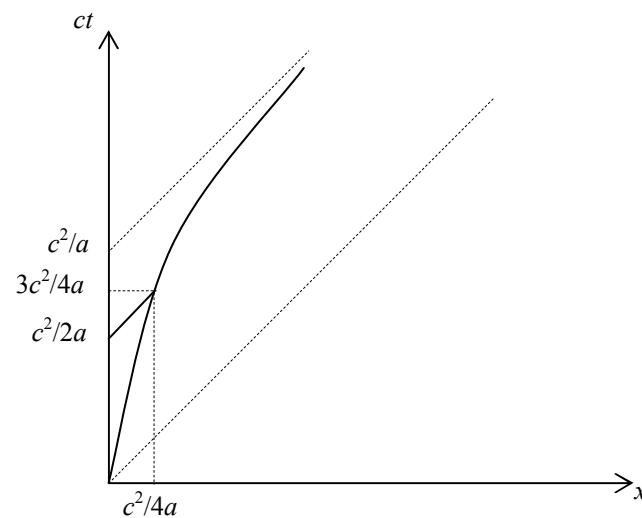
$$x(t) = c \left(t - \frac{c}{2a} \right) \Rightarrow t = \frac{3}{4} \frac{c}{a} .$$

The position and the velocity of the rocket at that moment are

$$x \left(\frac{3}{4} \frac{c}{a} \right) = \frac{c^2}{4a} \quad \text{and} \quad v_x \left(\frac{3}{4} \frac{c}{a} \right) = \frac{3}{5} c$$

respectively. In order to find T' , for each infinitely small time interval dt in Earth's reference frame we will have to add up the corresponding time interval dt' in the rocket's reference frame.

$$dt' = dt \sqrt{1 - \frac{v_x^2}{c^2}} = \frac{dt}{\sqrt{1 + \left(\frac{at}{c} \right)^2}} \Rightarrow T' = \int_0^{\frac{3}{4} \frac{c}{a}} dt' = \frac{c}{a} \ln 2 \approx 0.7 \frac{c}{a} .$$

g. 0.5p

h. 1p

From the Doppler Effect formula it follows that

$$\nu' = \nu_0 \sqrt{\frac{c - v_x}{c + v_x}} = \frac{1}{2} \nu_0 .$$

i. 1p

Applying once again the Doppler Effect formula, we get

$$\nu = \frac{1}{4} \nu_0 .$$



3rd Romanian Master of Sciences 2010

Physics – Theoretical Tour

NEWTONIAN COSMOLOGY

a. 1p

First of all, it is evident that $\mathbf{v}$ and $\mathbf{r}$ must have the same direction. Secondly, the expansion of any arbitrary two position vectors at any moment of time must preserve the already existing proportionality between them. That is,

$$\frac{r_1(t)}{r_2(t)} = \frac{r_1(t+dt)}{r_2(t+dt)} = \frac{r_1 + v_1 dt}{r_2 + v_2 dt} = \frac{v_1(t)}{v_2(t)} .$$

So Hubble's Law is

$$\vec{v}(t) = H(t) \cdot \vec{r}(t) .$$

b. 1p

Let A and B be two galaxies seen from a point in space, e.g. from Earth. According to Hubble's Law,

$$\vec{v}_A = H(t) \cdot \vec{r}_A ,$$

$$\vec{v}_B = H(t) \cdot \vec{r}_B .$$

By subtracting the two expressions, we get

$$(\vec{v}_B - \vec{v}_A) = H(\vec{r}_B - \vec{r}_A) .$$

So the relative velocity of galaxy B with respect to galaxy A is proportional to its relative position with respect to that galaxy, the proportionality factor being the same Hubble constant.

c. 0.5p

Assuming that

$$\vec{r}(t_0) = \vec{v}(t_0) \cdot t_0 ,$$

we get

$$\vec{v}(t_0) = H(t_0) \cdot \vec{v}(t_0) \cdot t_0 \Rightarrow t_0 = \frac{1}{H_0} .$$

d. 0.5p

$$\rho(t) \frac{4\pi r^3(t)}{3} = \text{const} \Rightarrow \rho(t) \cdot R^3(t) \cdot r_0^3 = \text{const} \Rightarrow \rho(t) = \frac{\rho_0}{R^3(t)} .$$

e. 0.5p

$$E(t) = \frac{mv^2(t)}{2} - G \frac{m\rho(t) \frac{4\pi}{3} r^3(t)}{r(t)} = \frac{mH^2(t)r^2(t)}{2} - \frac{4\pi Gm\rho(t)r^2(t)}{3} = \frac{mR^2(t)r_0^2}{2} \left(H^2(t) - \frac{8\pi G\rho(t)}{3} \right) .$$

f. 0.5p

If $\Omega > 1$, the expansion will eventually come to a halt and then the universe will start to shrink until it vanishes.

If $\Omega = 1$, the universe will keep on expanding, approaching infinity with zero recessional velocity.

If $\Omega < 1$, the universe will expand to infinity with nonzero recessional velocity.

g. 0.5p

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} \Rightarrow E(t) = \frac{mR^2(t)r_0^2H^2(t)}{2}(1-\Omega(t)) .$$

Since $E = \text{const}$, the sign of $1 - \Omega(t)$ does not change with time.

h. 0.5p

$$v(t) = H(t)r(t) \Rightarrow H(t) = \frac{1}{r(t)} \frac{dr}{dt} = \frac{1}{R(t)} \frac{dR}{dt} \Rightarrow E(t) = \frac{mr_0^2}{2} \left[\left(\frac{dR}{dt} \right)^2 - \frac{8\pi G \rho_0}{3R(t)} \right] .$$

$$\rho_0 = \Omega_0 \rho_{c0} = \frac{3\Omega_0 H_0^2}{8\pi G} \Rightarrow E(t) = \frac{mr_0^2}{2} \left[\left(\frac{dR}{dt} \right)^2 - \frac{\Omega_0 H_0^2}{R(t)} \right] = \frac{mr_0^2 H_0^2 (1 - \Omega_0)}{2} \Rightarrow$$

$$\left(\frac{dR}{dt} \right)^2 = \frac{\Omega_0 H_0^2}{R} - H_0^2 (\Omega_0 - 1) = H_0^2 \left(\frac{\Omega_0}{R} + 1 - \Omega_0 \right) .$$

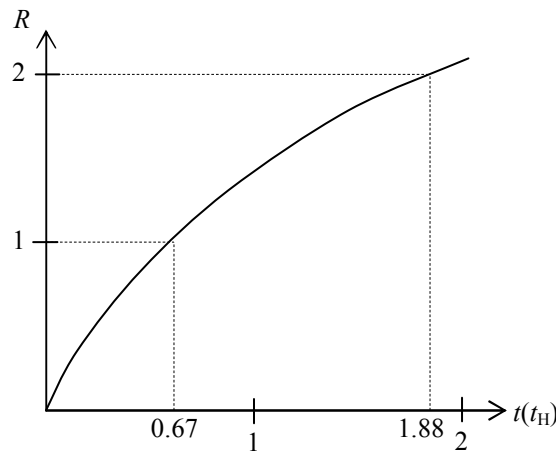
$$t \rightarrow 0 \Rightarrow R \rightarrow 0 \Rightarrow \frac{dR}{dt} \rightarrow \infty \Rightarrow RH \rightarrow \infty \Rightarrow 1 - \Omega \rightarrow 0 .$$

i. 0.5p

$$\frac{dR}{dt} = \frac{H_0}{\sqrt{R}} \Rightarrow \int \sqrt{R} dR = \int H_0 dt \Rightarrow \frac{2}{3} R^{\frac{3}{2}} = H_0 t \Rightarrow R(t) = \left(\frac{3}{2} \frac{t}{t_H} \right)^{\frac{2}{3}} .$$

Since $R_0 = 1$, we get $t_0 = 2/3 t_H$.

j. 0.5p



k. 0.5p

$$\frac{dR}{dt} = H_0 \sqrt{\frac{\Omega_0 - (\Omega_0 - 1)R}{R}} \Rightarrow \sqrt{\frac{R}{1 - \frac{\Omega_0 - 1}{\Omega_0} R}} dR = H_0 \sqrt{\Omega_0} dt .$$

Now

$$x = \frac{\Omega_0 - 1}{\Omega_0} R \Rightarrow \left(\frac{\Omega_0}{\Omega_0 - 1} \right)^{\frac{3}{2}} \sqrt{\frac{x}{1-x}} dx = H_0 \sqrt{\Omega_0} dt \Rightarrow \arcsin \sqrt{x} - \sqrt{x(1-x)} = \frac{H_0}{\Omega_0} (\Omega_0 - 1)^{\frac{3}{2}} t .$$

(We took into account the fact that $x(0) = 0$.) So

$$t(R) = \frac{\Omega_0}{H_0 (\Omega_0 - 1)^{\frac{3}{2}}} \left[\arcsin \sqrt{\frac{\Omega_0 - 1}{\Omega_0} R} - \sqrt{\frac{\Omega_0 - 1}{\Omega_0} R \left(1 - \frac{\Omega_0 - 1}{\Omega_0} R \right)} \right] .$$

l. 0.5p

$$\sqrt{\frac{\Omega_0 - 1}{\Omega_0}} R = \sin \frac{p}{2} \Rightarrow R = \frac{\Omega_0}{\Omega_0 - 1} \sin^2 \frac{p}{2} \Rightarrow$$

$$\begin{cases} R(p) = \frac{1}{2} \frac{\Omega_0}{\Omega_0 - 1} (1 - \cos p) ; \\ t(p) = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{\frac{3}{2}}} (p - \sin p) . \end{cases}$$

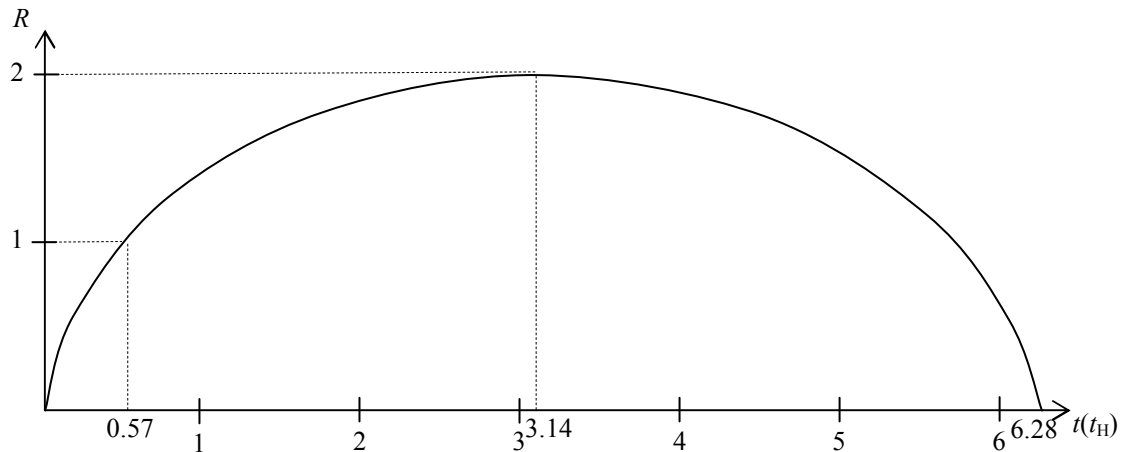
m. 0.5p

$$R(T) = 0 \Rightarrow 1 - \cos p = 0 \Rightarrow p = 2\pi \Rightarrow T = \pi \frac{\Omega_0}{(\Omega_0 - 1)^{\frac{3}{2}}} t_H .$$

n. 0.5p

$$R_{\max} = \frac{\Omega_0}{\Omega_0 - 1} \text{ for } p = \pi, \text{ i.e. at } t = T/2 .$$

o. 0.5p



p. 0.5p

$$\frac{dR}{dt} = H_0 \sqrt{\frac{\Omega_0 + (1 - \Omega_0) R}{R}} \Rightarrow \sqrt{\frac{R}{1 + \frac{1 - \Omega_0}{\Omega_0} R}} dR = H_0 \sqrt{\Omega_0} dt$$

Now

$$x = \frac{1-\Omega_0}{\Omega_0} R \Rightarrow \left(\frac{\Omega_0}{1-\Omega_0} \right)^{\frac{3}{2}} \sqrt{\frac{x}{1+x}} dx = H_0 \sqrt{\Omega_0} dt \Rightarrow -\operatorname{arcsinh} \sqrt{x} + \sqrt{x(1+x)} = \frac{H_0}{\Omega_0} (1-\Omega_0)^{\frac{3}{2}} t .$$

(We took into account the fact that $x(0) = 0$.) So

$$t(R) = \frac{\Omega_0}{H_0 (1-\Omega_0)^{\frac{3}{2}}} \left[-\operatorname{arcsinh} \sqrt{\frac{1-\Omega_0}{\Omega_0} R} + \sqrt{\frac{1-\Omega_0}{\Omega_0} R \left(1 + \frac{1-\Omega_0}{\Omega_0} R \right)} \right] .$$

q. 0.5p

$$\sqrt{\frac{1-\Omega_0}{\Omega_0}} R = \sinh \frac{p}{2} \Rightarrow R = \frac{\Omega_0}{1-\Omega_0} \sinh^2 \frac{p}{2} \Rightarrow$$

$$\begin{cases} R(p) = \frac{1}{2} \frac{\Omega_0}{1-\Omega_0} (\cosh p - 1) ; \\ t(p) = \frac{1}{2H_0} \frac{\Omega_0}{(1-\Omega_0)^{\frac{3}{2}}} (\sinh p - p) . \end{cases}$$

r. 0.25p

$$\lim_{p \rightarrow \infty} \frac{R(p)}{t(p)} = H_0 \sqrt{1-\Omega_0} \Rightarrow R(t) \propto \sqrt{1-\Omega_0} \frac{t}{t_H} .$$

s. 0.25p

