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1. Note that
   A: has dimensions of energy.
   B: has dimensions of length.
   C: is dimensionless.
   Now the Bohr radius will have a combination of B and C.

$$r_n = \frac{n^2}{2\pi}\frac{B}{C}$$

   The energy will have a combination of A and C.

$$E_n = -\frac{1}{2n^2}AC^2$$

   The Rydberg constant (of dimensions length inverse) will have a combination of B^{-1} and C.

$$R = \frac{C^2}{2B}$$

   Note: This problem can be done in number of ways.

2.

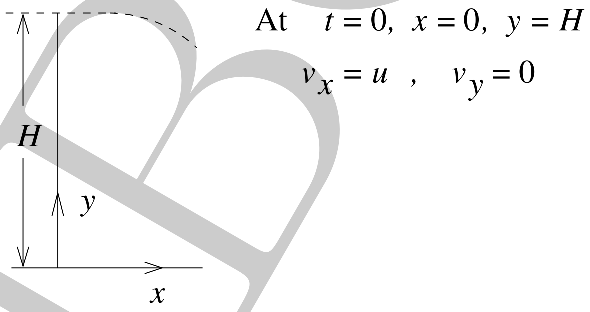

Figure 18: Problem 2 (a)

(a) See Fig.(18)

$$\begin{aligned}
\frac{dv_x}{dt} &= -cv_x \\
x &= \frac{u}{c}\left(1 - e^{-ct}\right) \qquad (39) \\
\frac{d\,v_y}{dt} &= -g - cv_y \\
\frac{dy}{dt} &= -\frac{g}{c}\left(e^{-ct} - 1\right) \\
\text{and} \qquad y &= H + \frac{g}{c^2} - \frac{g}{c}\left[\frac{1}{c}e^{-ct} + t\right] \qquad (40)
\end{aligned}$$

(b) From Eq. (39)

$$t = -\frac{1}{c}\ln\left(1-\frac{xc}{u}\right)$$

Substituting in Eq. (40),

$$y = H + \frac{g}{c^2} - \frac{g}{c}\left[\frac{1}{c}e^{\ln\left(1-\frac{xc}{u}\right)} - \frac{1}{c}\ln\left(1-\frac{xc}{u}\right)\right]$$

$$y = H + \frac{g}{c^2} - \frac{g}{c^2}\left(1-\frac{xc}{u}\right) + \frac{g}{c^2}\ln\left(1-\frac{xc}{u}\right)$$

$$y = H + \frac{gx}{cu} + \frac{g}{c^2}\ln\left(1-\frac{xc}{u}\right)$$

If c is small and the range is limited, e.g. $\frac{xc}{u} << 1$ then

$$y = H + \frac{gx}{cu} + \frac{g}{c^2}\left[-\frac{xc}{u} - \frac{x^2\,c^2}{2\,u^2} - \frac{x^3\,c^3}{3\,u^3} + \cdots\cdots\right] \tag{41}$$

$$y = H - \frac{g\,x^2}{2\,u^2} - \frac{g\,x^3\,c}{3\,u^3}$$

(c) The trajectory is foreshortened (see Fig. (19)).
Note $c = 0$ is without air resistance
and $c \neq 0$ is with air resistance.

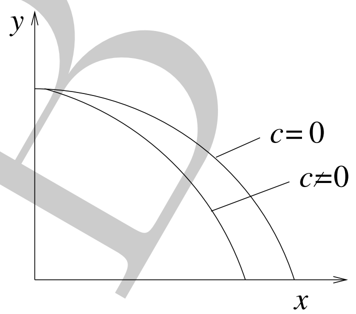

Figure 19: Problem 2 (c)

(d) When $y = 0$, from Eq. (40)

$$H + \frac{g}{c^2}\left[1 - e^{-ct} - ct\right] = 0$$

For $H = 500$ m, $c = 0.05\ \mathrm{s}^{-1}$, $g = 10\ \mathrm{m{\cdot}s}^{-2}$

$$t = 10 + 1.1 = 11.1\ \mathrm{s}$$

The above answer is obtained by iterative analysis. You may verify that the approximation in Eq. (41) is valid.

3. (a) Free Body Diagram, see Fig. (20)

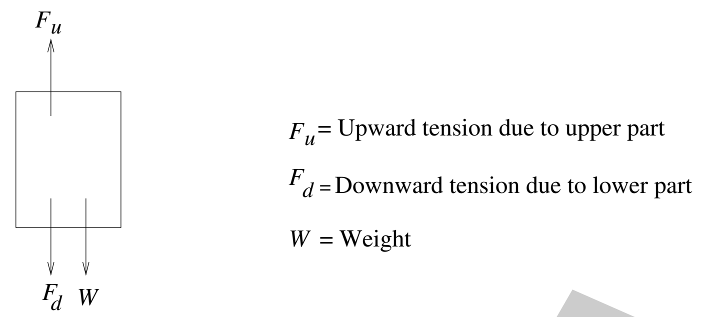

Figure 20: Problem 3 (a)

(b)

$$F_U - F_D = W - F_C$$

$$A\,(dT) = \frac{GM\rho\,A\,dr}{r^2} - \omega^2\,r\,\rho\,A\,dr$$

$$\frac{dT}{dr} = GM\rho\left[\frac{1}{r^2} - \frac{r}{R_g^3}\right] \quad \text{(using Kepler's third law)} \tag{42}$$

(c) Integrating Eq. (42)

$$\int_a^b dT = \int GM\rho\left(\frac{1}{r^2} - \frac{r}{R_g^3}\right) dr$$

$$T_b - T_a = GM\rho\left[-\frac{1}{r} - \frac{r^2}{2\,R_g^3}\right]\Bigg|_a^b \tag{43}$$

For $r \to R_g$ and $T(R) = 0$

$$T(R_g) = GM\rho\left[-\frac{3}{2\,R_g} + \frac{R^2}{2\,R_g^3} + \frac{1}{R}\right] \tag{44}$$

For $r \to R_g$ and $T(H) = 0$

$$T(R_g) = GM\rho\left[-\frac{3}{2\,R_g} + \frac{H^2}{2\,R_g^3} + \frac{1}{H}\right] \tag{45}$$

Equating Eqs. (44) and (45) we obtain

$$\begin{aligned} H &= \frac{R}{2}\left[\sqrt{1 + \frac{8\,R_g^3}{R^3}} - 1\right] \\ &= 1.51 \times 10^5 \text{ km} \end{aligned}$$

(d) See Fig. (21)

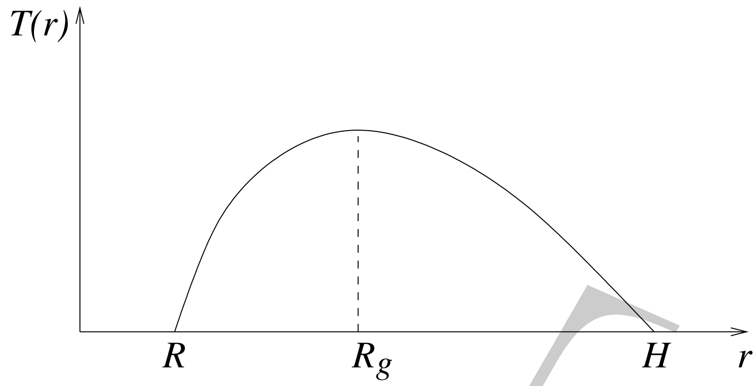

Figure 21: Problem 3 (d)

(e) Maximum stress will be at $r = R_g$
Using Eq. (44)

$$T(R_g) = 379 \ \text{GPa}$$

Steel has tensile strength 6.37 GPa which is less than 379 GPa. Hence it will not be feasible.

4.

$$T_0 = \frac{T_1 + T_2}{2}, \ \Delta = T_2 - T_1, \ \delta = T'' - T'$$

$T_1 = 270\text{K}$, $T_2 = 298\text{K}$

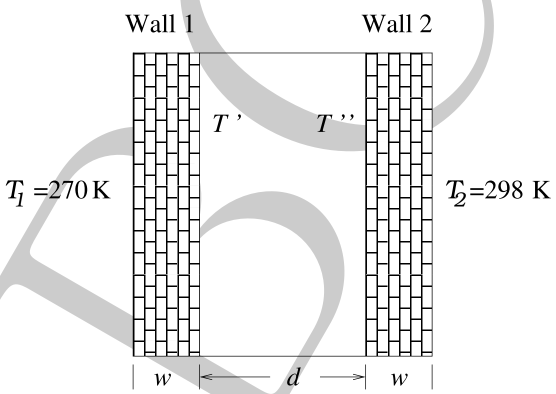

Figure 22: Problem 4

(a) For wall 1, $q_w = \frac{k_w}{w}(T_2 - T'')$

For wall 2, $q_w = \frac{k_w}{w}(T' - T_1)$

(b)

$$\Delta - \delta = T_2 - T'' + T' - T_1$$

For stationary process,

$$T_2 - T'' = T' - T_1 \tag{46}$$

$$\Delta - \delta = 2(T_2 - T'')$$

Hence,

$$q_w = \frac{k_w}{w} \frac{(\Delta - \delta)}{2}$$

(c)
$$E_1 = \epsilon\sigma T'^4 \ + \ (1-\epsilon)E_2$$

and
$$E_2 = \epsilon\sigma T''^4 \ + \ (1-\epsilon)E_1$$
$$\begin{aligned} q_r &= E_2 - E_1 \\ &= \frac{\epsilon\,\sigma}{(2-\epsilon)}\,(T''^4 - T'^4) \end{aligned}$$

(d) Using Eq. (46) and given set of equations,
$$T'' = T_0 + \frac{\delta}{2}, \quad T' = T_0 - \frac{\delta}{2} \tag{47}$$

also,
$$\begin{aligned} T''^4 - T'^4 &= (T''^2 - T'^2)(T''^2 + T'^2) \\ &= 2\delta\, T_0\,[(T'' + T')^2 - 2T''\,T'] \end{aligned}$$

putting values from Eq. (47)
$$= 4\delta\, T_0^3 \left[1 + \left(\frac{\delta}{2\,T_0}\right)^2\right]$$

again, $\quad q_r = q_w$
$$\frac{\epsilon\,\sigma}{(2-\epsilon)}\,(T''^4 - T'^4) = \frac{k_w}{w}\;\frac{(\Delta-\delta)}{2}$$

since, $\quad (\delta^2 << T_0^2)$
$$\frac{\epsilon\,\sigma}{(2-\epsilon)}\,4\,\delta\,T_0^3 = \frac{k_w}{w}\;\frac{(\Delta-\delta)}{2}$$
$$1 - \frac{\delta}{\Delta} = 8\,c\,T_0^3\,\frac{\delta}{\Delta}$$

where,
$$c = \frac{\sigma\,\epsilon\,w}{k_w\,(2-\epsilon)}$$
$$\delta = \frac{\Delta}{1 + 8cT_0^3}$$

Now,
$$q_r = q_w = \frac{k_w}{w}\,\frac{(\Delta-\delta)}{2} \ = \frac{k_w}{w}\left(\frac{\Delta\,4c\,T_0^3}{1+8c\,T_0^3}\right)$$

(e)
$$c = 6.44\times10^{-10},\ w = 0.01\ \text{ m},\ k_w = 0.72\ \text{W}\cdot\text{m}^{-1}\cdot\text{K}^{-1}$$
$$T_0 = 284\ \text{K},\ \Delta = 28\,\text{K}$$
$$q_r = 107.22\ \text{W}\cdot\text{m}^{-2}$$

(f)

$$q_{cv} = \frac{N_u\,K_a}{d}\,(T'' - T') = \frac{N_u\,K_a}{d}\,\delta$$

$$q_{cv} = q_w$$

gives,

$$\delta = \frac{\Delta}{1 + \dfrac{2\,w\,k_a\,N_u}{k_w\,d}}$$

$$q_{cv} = \frac{N_u\,k_a\,\Delta\,k_w}{k_w\,d + 2\,w\,k_a\,N_u}$$

(g) Ignoring $2wk_a\,N_u$
$q_{cv} \simeq 46.5\ \ \mathrm{W{\cdot}m^{-2}}$

(h)

$$q_{cd} = \frac{k_s}{d}(T'' - T') = \frac{k_s\delta}{d}$$

$$q_{cd} = q_w$$

From part (b),

$$\frac{k_w}{2\,w}(\Delta - \delta) = \frac{k_s\,\delta}{d}$$

which gives,

$$\delta = \frac{k_w\,\Delta\,d}{2\,wk_s + k_w\,d}$$

Hence,

$$q_{cd} = \frac{k_s\,k_w\,\Delta}{2\,wk_s + k_w\,d}$$

(i) $k_s = 0.05\ \mathrm{W{\cdot}m^{-1}K^{-1}}$, $k_w = 0.72\ \mathrm{W{\cdot}m^{-2}}$
$w = 0.01$ m, $d = 0.1$ m
$q_{cd} = 13.8\ \mathrm{W{\cdot}m^{-2}}$

(j) Since,

$$q_{cd}\ <\ q_{cv}\ <\ q_r$$

Hence sheathing material is best for insulation.

5. See the brief solution.

6. (a) The force on one capacitor plate due to the other is

$$F_e = \frac{Q_1^2}{2A\epsilon_0}$$

$$F_e = \frac{C_1^2 V_0^2\cos^2(2\pi f t)}{2\pi a^2\epsilon_0}$$

Time - averaged force is

$$\langle F_e\rangle = \frac{C_1^2\,V_0^2}{4\,\pi\epsilon_0\,a^2} \qquad (48)$$

(b) Charge on Capacitor C_2: $Q_2 = C_2 V_0 \cos(2\pi f t)$

$$i_2 = -C_2 V_0 \, 2\pi f \sin(2\pi f \, t)$$

(note: wire has negligible resistance.)
Force on one ring due to the other

$$F_m = i_2 l B$$

Hence,

$$F_m = \frac{\mu_0 \, b}{h}(-C_2 V_0 2\pi f \sin(2\pi f t))^2$$

Time- averaged force is

$$\langle F_m \rangle = \frac{\mu_0 \, b}{2\, h} C_2^2 V_0^2 (2\pi f)^2 \tag{49}$$

(c) Equating Eqs. (48) and (49)

$$\langle F_e \rangle = \langle F_m \rangle$$

$$\text{and noting } \; c^2 = \frac{1}{\mu_0 \, \epsilon_0}$$

We obtain

$$c = (2\pi)^{3/2} \, a \left(\frac{b}{h}\right)^{1/2} \frac{C_2}{C_1} f \tag{50}$$

(d) From Eq. (50) and given constants.

$$c = 2.99 \times 10^8 \text{ m}\cdot\text{s}^{-1}$$

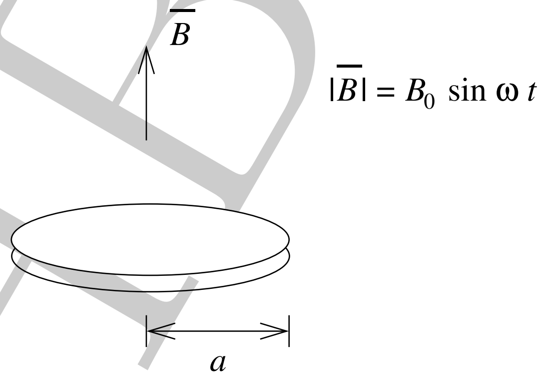

Figure 23: Problem 7

7. (a) Flux : $\phi = B_0 \sin(\omega t)\, \pi a^2 N$

$$\varepsilon = -\frac{d\phi}{dt} = -N\pi a^2 B_0 \, \omega \cos \omega t$$

From Kirchhoff's Law,

$$i\, R + L\,\frac{di}{dt} = -N\pi a^2 B_0 \, \omega \cos \omega t \tag{51}$$

(b) Take $i = Rl(i_0 e^{j\omega t})$ $(j^2 = -1)$
Substituting the complex form in Eq. (51), we obtain,

$$i_0 = \frac{N\,\pi\,a^2 B_0\,\omega(R - j\omega L)}{R^2 + \omega^2\,L^2}$$

This implies,

$$i = \frac{N\,\pi\,a^2 B_0\,\omega(R\cos\omega t + \omega L\sin\omega t)}{R^2 + \omega^2\,L^2}$$

(c) The elemental force

$$d\overline{F} = i\,d\bar{l} \times \overline{B}$$

is directed radially in. Substituting,

$$\frac{dF}{dl} = -\frac{NB_0^2\,\pi a^2\omega}{R^2 + \omega^2 L^2}(R\,\sin\omega t\,\cos\omega t + \omega\,L\sin^2\omega t)$$

Time - averaged compressional force

$$\left.\frac{dF}{dl}\right|_{av} = -\frac{NB_0^2\,\pi a^2\omega^2 L}{2(R^2 + \omega^2 L^2)} \quad (52)$$

$$\left.\frac{dF}{dl}\right|_{osc} = -\frac{NB_0^2\,\pi a^2\omega}{2(R^2 + \omega^2 L^2)}(R\sin 2\omega t - \omega L\cos 2\omega t) \quad (53)$$

Net force on ring is zero by symmetry.

(d) From Eq. (52)

$$\begin{aligned}\left.\frac{dF}{dl}\right|_{av} &= -\frac{\pi(10^{-2})\,10^6\,10^{-1}}{2(10^2 + 10^4)} \\ &\simeq -\frac{\pi}{2}10^{-1}N \\ &= 1.55\ \text{N}\cdot\text{m}^{-1}\end{aligned}$$

(e) i. From Eq. (53) the oscillating force has a frequency of 2ω and hence the frequency of the sound is 120 Hz.

ii. The inclusion of the capacitor will result in
$\omega L \to \omega L - \dfrac{1}{\omega C}$
$\therefore$ the compressional force is lessened and may even become negative, i.e. tensile and outward.