THEORETICAL COMPETITION

January 14, 2025

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. You can use your own calculator for numerical calculations. If you don’t have one, please ask for it from Olympiad organizers.
3. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
4. Use only the front side of ***Writing sheets*.** Write only inside the boxed area.
5. Each task should be solved starting from a new page of the ***Writing sheets***
6. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.

Problem 1 (10.0 points)

This problem consists of three independent parts.

Problem 1.1 (3.0 points)

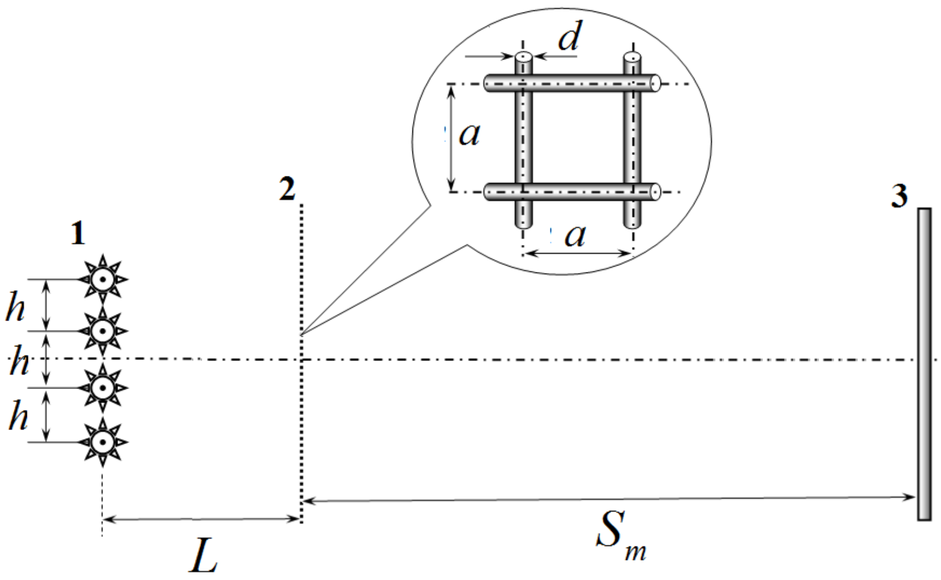
An interplanetary spacecraft takes off from the Earth's surface with the second cosmic velocity $v_{2E} = 11.2$ km/s. Moving along a ballistic trajectory, the spacecraft, due to calculation errors, is captured by the Moon's gravity and falls onto its surface. Find the possible minimum v_{min} and maximum v_{max} collision velocities of the spacecraft with the Moon. The ratio of the mass M_E of the Earth to the mass M_M of the Moon is $M_E/M_M = 80.0$; the ratio of the radius R_E of the Earth to the radius R_M of the Moon is $R_E/R_M = 3.70$; the distance from the Earth to the Moon is $r = 60.0R_E$. Neglect the influence of the Sun.

Problem 1.2 (4.0 points)

A fully deflated balloon with a maximum volume of $V_0 = 22.4\ l$, made of soft, non-stretchable rubberized fabric, is connected through a valve to a gas holder, which is a very large vessel with nitrogen. Assume that the nitrogen pressure in the gas holder is constant and equal to $P = 5.00$ atm, the ambient air pressure is $P_0 = 1.00$ atm, and the nitrogen temperature in the gas holder is equal to the ambient air temperature $T_0 = 273$ K. The gas holder valve opens and, after the balloon is quickly filled with nitrogen to its maximum volume and the nitrogen pressure in the balloon becomes equal to the pressure in the gas holder, the valve is closed. Find the number of moles of nitrogen ν in the balloon. Neglect heat exchange with the ambient air during the time the balloon is filled with nitrogen.

Problem 1.3 (3.0 points)

The light sources of flashlight 1 are four identical white LED bulbs located on one line at a distance of $h = 4.90$ mm from each other. The bulbs can be considered point light sources. At a distance of $L = 10.0$ cm, parallel to them, there is a metal grid 2 made of thin wire with a diameter of $d = 0.20$ mm. The grid cells are square, whose side is equal to $a = 1.00$ mm. Calculate at what distances S_m $(m = 1,2 ...)$) the screen 3 should be located so that the grid image on it is the clearest and sharpest.



Problem 2. First estimate of the age of the Earth (10.0 points)

Fourier's law of thermal conductivity

Thermal conductivity characterizes the ability of a body to transfer thermal energy from one point to another if there is a temperature difference between them. Let us consider a flat homogeneous plate with a cross-sectional area S and a thickness dx, whose surfaces have the temperature difference dT. The phenomenon of thermal conductivity in a one-dimensional stationary case is described by Fourier's law, which has the following form:

$$dQ = -\kappa \frac{dT}{dx} S dt,$$

where dQ is the amount of heat transferred during time dt through the plate surface in the direction of the normal coinciding with the direction of decreasing temperature. The constant κ is called the thermal conductivity coefficient and depends on the plate material, and the value $j_Q = \frac{1}{S}\frac{dQ}{dt}$ refers to the modulus of the heat flux, which is a vector directed towards the heat transfer.

2.1 The left end of the rod, enclosed in a heat-insulating shell, is maintained at a temperature of $t_1 = 25°\mathrm{C}$, whereas its right end is kept at a temperature of $t_2 = 100°\mathrm{C}$. The rod itself consists of two parts of the same length, but different thermal conductivities. The thermal conductivity coefficient of the left part is twice as high as the thermal conductivity coefficient of the right part. Calculate the temperature t of the contact surface of the rod parts.

2.2 A rod of length l with a heat-insulated side surface consists of a material whose thermal conductivity coefficient varies along according to the law $\kappa = a/T^2$, where a is a certain constant. The ends of the rod are maintained at temperatures T_1 and T_2. Determine the dependence of temperature $T(x)$ on the coordinate x, which is the distance from the rod end with temperature T_1.

2.3 Estimate the time it takes for an ice crust of thickness $h = 4.00$ cm to form on the surface of a lake at an ambient air temperature of $t = -20.0°\mathrm{C}$. The thermal conductivity coefficient of ice is $\kappa = 2.20\ \mathrm{W/(m\cdot K)}$, its density is $\rho = 900\ \mathrm{kg/m^3}$, and the specific melting heat is $\lambda = 3.33\cdot 10^5\ \mathrm{J/kg}$.

Electrothermal analogy

Electrothermal analogy is a method for calculating thermal systems based on replacing thermal quantities with their electrical analogues. The method is based on the identity of the mathematical apparatus of thermal physics and electrical engineering: the propagation of heat and electric current are described by the same differential equations, while measuring the electrical characteristics of real objects is much simpler.

2.4 Find Below a table of correspondence between thermal and electrical quantities. Thermal quantities are written out explicitly, and you need to transfer the table to your solution and write the corresponding electrical quantities in front of them: electric current density j_E, specific conductivity σ, electric charge q, electric field potential φ.

Thermal quantity	Electrical quantity
temperature T	
amount of heat Q	
heat flux j_Q	
thermal conductivity coefficient κ	

2.5 A cube consists of identical thin rods with thermally insulated side surfaces. Two vertices of the cube A and B are maintained at constant temperatures $t_A = 20°\mathrm{C}$ and $t_B = 120°\mathrm{C}$ respectively. Determine the temperature t_C at vertex C, see Figure 2.1 below.

2.6 A thin heat-conducting sphere of radius R is made of a material with the wall thickness $h \ll R$. Thermal power P is supplied to point A of the sphere, and the same thermal power is removed at point B of the sphere. Neglecting heat losses, determine the modulus of the heat flux at point C of the sphere, see Figure 2.2 below (O is the sphere center, $AO \perp OB, OC \perp OA, OC \perp OB$).

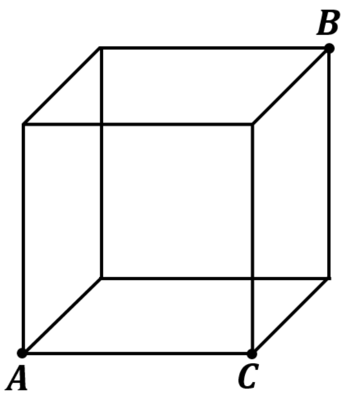


Figure 2.1

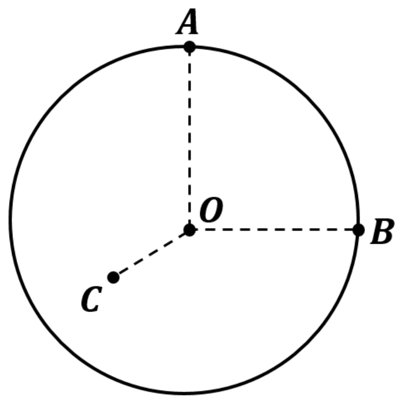


Figure 2.2

The first estimate of the Earth age

In the 18th century, there lived a famous naturalist, Georges-Louis Leclerc, Count de Buffon (1707-1788), who was a member of the Paris Academy of Sciences, the French Academy, a member of the Royal Society of London, and also a foreign honorary member of the St. Petersburg Academy of Sciences. He published his works in a multi-volume work, Natural History, in which he attempted to systematize the knowledge gained to that time in a variety of fields of science, and also expressed a number of advanced ideas. His interests included geology, and he was the first to formulate the problem of determining the age of the Earth. In particular, Leclerc developed a hypothesis about the origin of the planets through the collision of celestial bodies, which should have led to their heating and subsequent cooling. It was through the process of temperature change over time that the first estimate of the Earth age in history was obtained.

Of course, Leclerc did not know the laws of physics in their modern form and did not have a sufficient command of the mathematical apparatus, but he had a keen physical sense.

2.7 Let a teapot have a spherical shape with a radius $r_0 = 15.0$ cm and after heating water in it to the boiling point, it cools down to the room temperature in a time of $\tau_0 = 6.00$ h. Let us imagine a hypothetical situation: the teapot is proportionally increased to the size of the Earth, the radius of which is $R_E = 6400$ km. Estimate the cooling time τ of such a hypothetical "teapot" under the same external conditions.

Unlike Count de Buffon, we are familiar with the modern physics laws, so let us consider the simplest model of the Earth, whose radius is equal to $R_E = 6400$ km. Inside the Earth there is a core with a radius of $R_0 = 3500$ km, whose substance perfectly conducts heat ($\kappa = \infty$) and has a specific heat capacity of $c = 500$ J / (kg · K) at a density of $\rho = 5.46\ \mathrm{g/cm^3}$. Outside the core, right up to the Earth's surface, is the mantle, whose substance has zero specific heat capacity and a thermal conductivity coefficient of $\kappa = 50.3\ \mathrm{W/(m \cdot K)}$. Neglect the influence of the Earth's crust, and assume that the Sun has no thermal effect on the Earth, which can be considered an absolutely black body being in a vacuum.

2.8 At the present moment in time, the Earth's surface temperature is known to be $T = 300$ K. Calculate the temperature T_0 of the Earth's core within the framework of the presented model.

2.9 Calculate the temperature of the mantle T_H at a depth of $H = 2.00$ km from the Earth's surface.

2.10 Calculate how much the temperature ΔT of the Earth's surface differs from what it was $\tau = 1.00 \cdot 10^6$ years ago.

Problem 3. Dirac's monopole (10.0 points)

Introduction

A monopole is a hypothetical elementary particle which, unlike all known magnets, is a unipolar magnet. The hypothesis of the existence of magnetic monopoles was put forward by the English physicist Paul Dirac in 1931, and a number of theoretical models have been developed to date of the electromagnetic field in the presence of magnetic charges (monopoles). Note that numerous attempts to experimentally detect magnetic monopoles have been unsuccessful.

Let us accept the hypothesis of the existence of point-like monopoles with a magnetic charge q_m. By analogy with Coulomb's law, it is assumed that the magnetic field created by the monopole is central, and the modulus of the induction vector B at a distance r from the particle is determined by the formula

$$B = \frac{\mu_0 q_m}{4\pi r^2},$$

where μ_0 is the magnetic constant.

3.1 The dimension of magnetic charge q_m in the SI system of units is: $[q_m] = \mathrm{A}^{x_1} \cdot m^{x_2} \cdot kg^{x_3} \cdot s^{x_4}$, where x_1, x_2, x_3, x_4 are some integers. Find all x_1, x_2, x_3, x_4.

A ring of radius $R = 2.50$ cm made of thin superconducting wire with thickness $\delta = R/100$ is used as a detector of varying magnetic fields. The induced electric current arising in the ring can be recorded and its strength can be measured. When particles with magnetic fields pass through the ring, conclusions about their magnetic field structures can be drawn from the character of the ring current variations. It is known that the inductance of a thin ring is determined by the formula

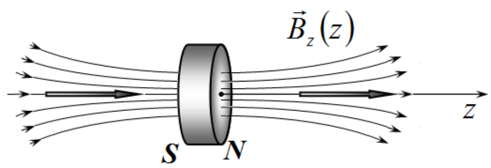
$$L = \mu_0 R \left(\ln\frac{16R}{\delta} - 2\right).$$

3.2 Calculate the inductance L of the superconducting ring.

Classical description

Assume that the motion of an electron in a superconductor obeys the laws of classical physics, and its mass can be neglected. This part of the problem requires drawing of schematic graphs. You are not required to strictly calculate the functions being analyzed, you are not required to indicate any numerical characteristics, it is enough to show its qualitative behavior: domains of increase and decrease, the presence of maxima and minima, the presence of discontinuity points, behavior with arguments tending to $\pm\infty$. You can construct these graphs even if you have not obtained the exact form of functional dependencies, such drawings will be graded as well.

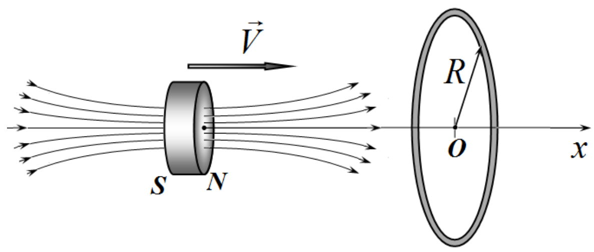
In the first experiment, a thin disk magnet is used, whose thickness is small compared to its radius. We align the z -axis with the axis of the magnet, and place the origin at its center. It can be shown that the cross-sectional average axial component B_z of the magnetic field induction vector on the axis of the magnet depends on the z -coordinate according to the law



$$B_z(z) = B_0 \frac{a^3}{(z^2+a^2)^{3/2}},$$

where B_0, a are known positive constants.

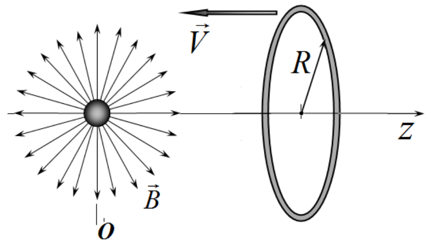
Let the disk magnet move with a constant velocity $\vec{V}$ along the axis of the superconducting ring, such that its axis always coincides with the axis of the ring. Let us consider this experiment in a frame of reference associated with the ring. We direct the coordinate axis Ox along the axis of the ring, its origin is coincident with the ring center, so that the coordinate of the magnet x changes from $-\infty$ to $+\infty$.



3.3 Find the dependence of the induction EMF $\varepsilon_{ind}(x)$, arising in the ring when the magnet moves, on the magnet coordinate x. Plot a schematic graph of this dependence.

3.4 Find the dependence of the current in the ring $I(x)$ on the coordinate of the magnet x, if at the initial moment of time the current was zero. Plot a schematic graph of this dependence.

3.5 Find the current in the ring I_f after the magnet flies away through the ring.



Let us now consider the second experiment on the passage of a monopole through the superconducting ring. Let the monopole move with a constant velocity $\vec{V}$ along the axis of the superconducting ring. In contrast, we consider this experiment in the reference frame associated with the monopole. Let us direct the coordinate axis Oz along the axis of the ring, its origin is coincident with the monopole, so that the coordinate of the ring z changes from $+\infty$ to $-\infty$.

3.6 Find the dependence of the Lorentz force $F_L(z)$, acting on a free electron in the ring from the magnetic field of the monopole, on the coordinate of the ring z.

3.7 Using the obtained dependence $F_L(z)$, find the EMF of induction in the ring $\varepsilon_{ind}(z)$ on the coordinate of the ring z. Plot a schematic graph of this dependence.

3.8 Find the dependence of the current in the ring $I(z)$ on the ring coordinate z if at the initial moment of time there was no current in the ring. Plot a schematic graph of this dependence.

3.9 Find the current in the ring I_f after the monopole flies away through the ring.

Quantization of magnetic flux in superconductors

Quantum theory shows that free electrons in a superconductor combine into bound pairs that move as a single unit (such pairs are called Cooper pairs). To describe their motion, one can use a semiclassical approximation: Newton's equations are supplemented with Bohr's quantization rule, which in this case states that the angular momentum M of a pair relative to the ring center can only take integer values that are multiples of the reduced Planck constant $\hbar$.

3.10 Express the change in the angular momentum of an electron pair ΔM moving along a circular superconducting ring through the change in the magnetic flux of the external magnetic field through the ring $\Delta\Phi$.

3.11 Find the minimum possible change in the magnetic flux of the external field through the ring, which is called the quantum of magnetic flux φ_0 and calculate its numerical value.

3.12 Find the change in current in the ring ΔI_0 when the magnetic flux through it changes by 1 quantum φ_0.

Estimating the Dirac Monopole Mass

By applying the quantization rule to the total magnetic flux generated by the Dirac monopole, we can estimate the magnetic charge and its mass. A Dirac monopole is a monopole with the minimum possible magnetic charge q_{m0}.

Theoretical physics uses the following model of an electron: it is a spherical particle with a charge uniformly distributed over the surface of a sphere whose radius is called the classical radius of an electron. It is also assumed that the rest mass of an electron is completely determined by the energy of its electrostatic field.

3.13 Find the numerical value of the minimum magnetic charge of a monopole q_{m0}.

3.14 Find the change in current ΔI in the superconducting ring if a Dirac monopole flies through it.

3.15 Express the classical radius of an electron r_0 through its mass m_e, charge e and fundamental physical constants. Calculate its numerical value.

3.16 Assuming that the radius of the monopole is equal to the classical radius of the electron, and its rest mass is determined by the energy of its magnetic field, find the mass ratio of the Dirac monopole m_m and of the electron m_e and calculate its numerical value.

Mathematical hint for the theoretical competition

The following formulas may be useful:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$ is a fixed number, C refers to an arbitrary constant;

$\int \frac{dx}{x} = \ln|x| + C$, where C stands for an arbitrary constant;

$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C$, where a – constant, C – arbitrary constant;

$(1+x)^\gamma \approx 1 + \gamma x + \frac{\gamma(\gamma-1)}{2}x^2$, for $|x| \ll 1$ and any value of γ;

$\tan x \approx \sin x \approx x$, for $|x| \ll 1$;

$\ln(1+x) \approx x$, for $|x| \ll 1$.

List of physical constants

Speed of light in vacuum $c = 3.00 \cdot 10^8$ m/s;

Reduced Planck constant $\hbar = 1.05 \cdot 10^{-34}$ J·s;

Magnetic constant $\mu_0 = 4\pi \cdot 10^{-7}$ H·m;

Electric constant $\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m;

Stefan-Boltzmann constant $\sigma = 5.67 \cdot 10^{-8}\ W/(m^2 \cdot K^4)$;

Elementary charge $e = 1.60 \cdot 10^{-19}$ C;

Electron mass $m_e = 9.11 \cdot 10^{-31}$ kg.