THEORETICAL COMPETITION

January 9, 2024

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with *Writing sheet* and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the *Writing sheets*. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of *Writing sheets.* Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country**), your student code (**Student Code**), the question number (**Question Number**), the progressive number of each sheet (**Page Number**), and the total number of *Writing sheets* used (**Total Number of Pages**). If you use some blank *Writing sheets* for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used *Writing sheets* in order.
    - The sheets you do not wish to be evaluated.
    - Unused sheets.
    - The printed problems.

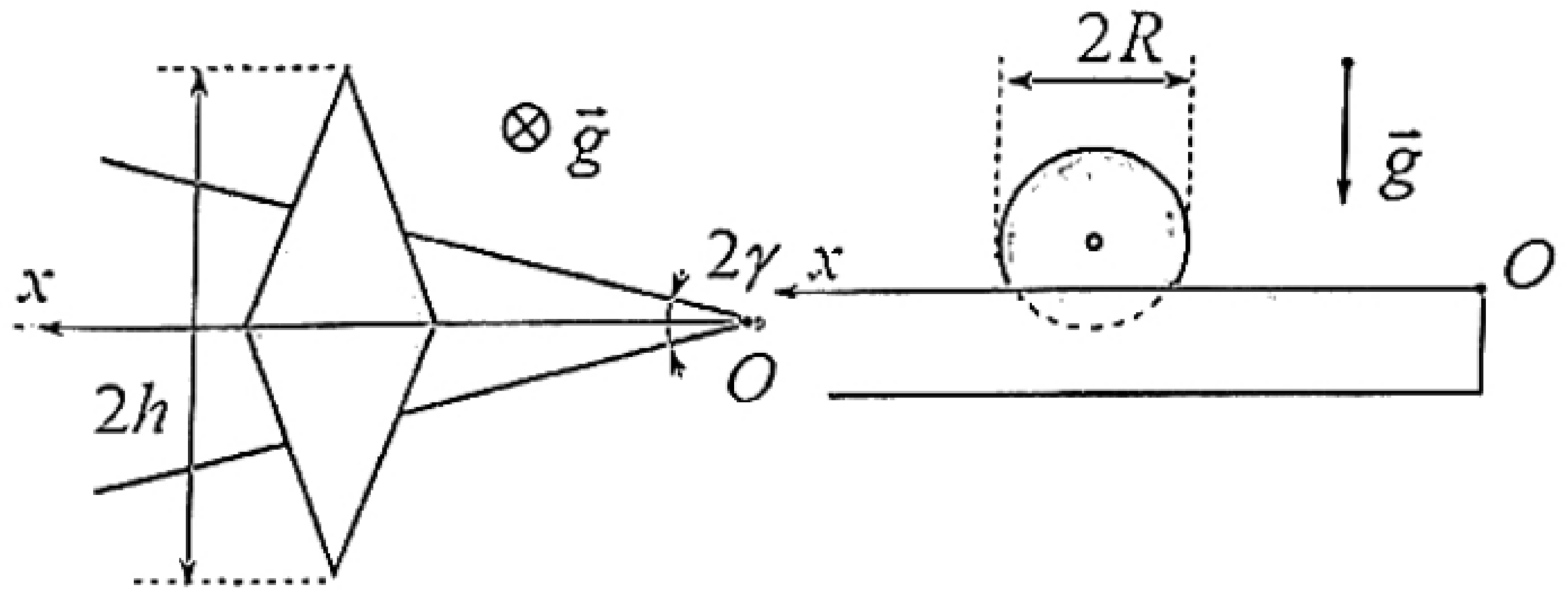
   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10.0 points)

This problem consists of three independent parts.

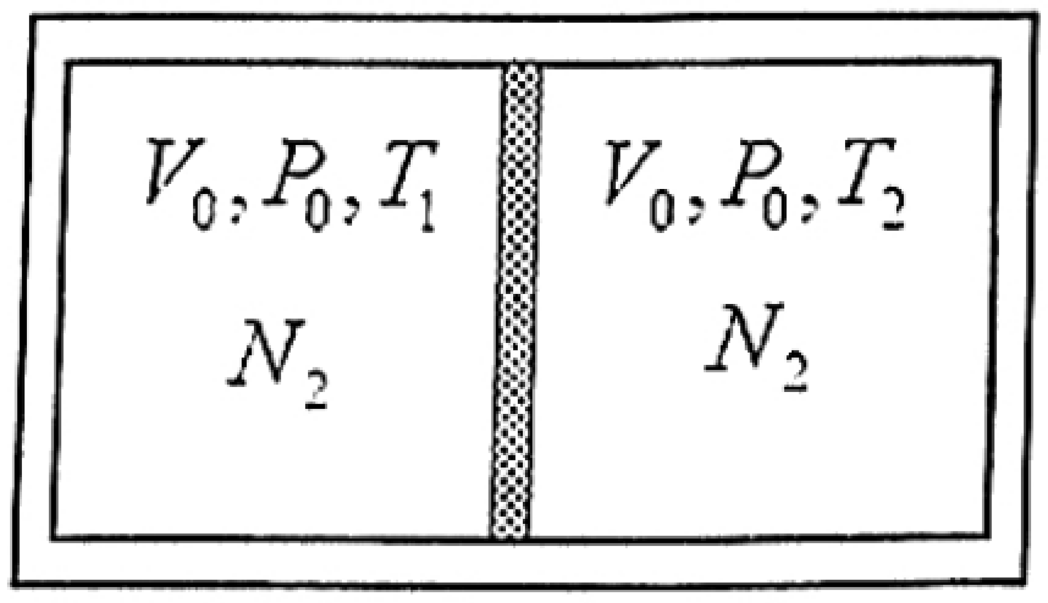
Problem 1.1 (3.0 points)

On two identical horizontal V-shaped slats (see figure below), forming a dihedral angle of 2γ ($\tan\gamma = 0.100$), there is a solid bicone with a base diameter of $D = 2R = 8.00$ cm and a distance between the vertices of $2h = 20.0$ cm, representing two identical homogeneous cones, rigidly fastened by their bases. First, the bicone is held so that its center of mass is right above the vertex of the dihedral angle, symmetrically relative to its bisectoral plane. Then it is released and it begins to roll along the slats without slipping. Find the dependence of the bicone speed of translational motion of the $v(x)$ on the x coordinate measured horizontally from the vertex of the dihedral angle to the center of mass of the bicone. Calculate the speed of the bicone v_0 at $x_0 = 50.0$ cm. Determine the maximum angular velocity ω_{max}, that the bicone is to reach during its motion. Consider that the height of the slats exceeds the radius of the cone bases, and the acceleration of gravity is $g = 9.80$ m/c^2.



Problem 1.2 (4.0 points)

A thermally insulated vessel with a volume of $2V_0 = 2.00$ l is divided by a thin weightless movable heat-conducting partition into two equal parts, each containing nitrogen N_2 at a pressure $P_0 = 100$ kPa. The initial temperature of nitrogen in one half of the vessel is $T_1 = 240$ K, and it is $T_2 = 360$ K in the other. As a result of heat exchange, the partition begins to move slowly without friction. 1) Determine the amount of heat Q that will be exchanged between parts of the vessel by the time complete thermodynamic equilibrium is established; 2) Calculate the minimum work A' that must be done to return the partition to its original position. Neglect the heat capacities of the vessel and partition.



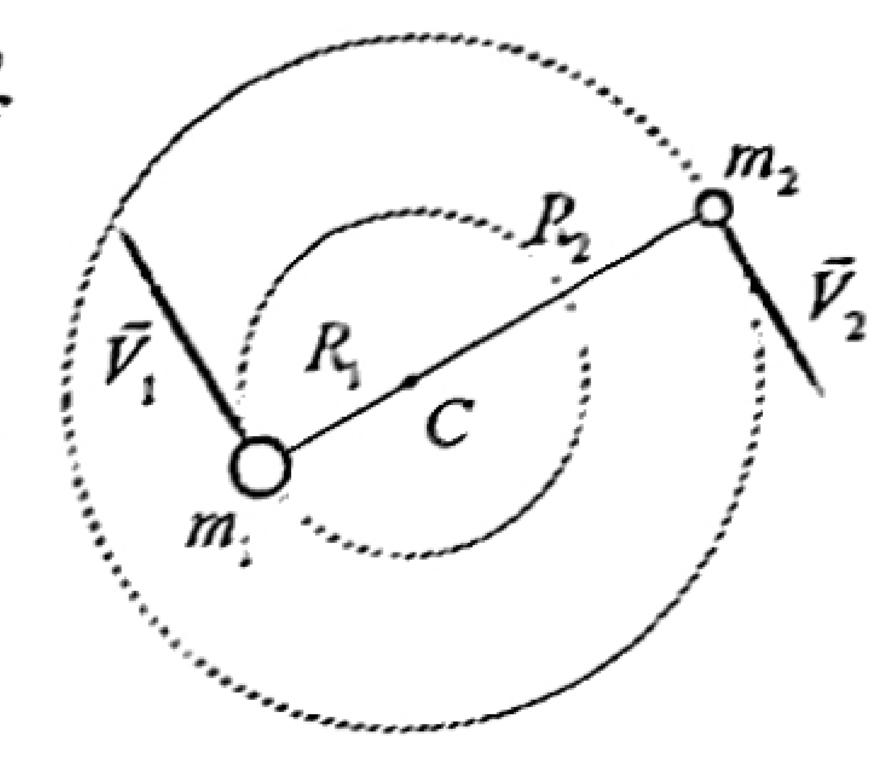
Problem 1.3 (3.0 points)

An electric motor connected to a battery lifts a load of mass m at a constant speed to a certain height h in time $t_1 = 1.00$ min. In this case, the load is tied to a thread that is wound on a uniformly rotating electric motor shaft. The time it takes to lift two loads with a total mass of $2m$ to the same height turns out to be $\Delta t = 7.00$ s longer. 1) Find the maximum number n_{max} of loads of the same mass m that such an electric motor can lift; 2) Calculate the time t_{max} that the electric motor is to require to uniformly lift the maximum number of loads n_{max} to the height h. Consider that the torque of an electric motor is proportional to the current flowing through its winding.

Problem 2. Lagrange points (10.0 points)

Two body problem

Consider the two-body problem. Two massive bodies with masses m_1, and $m_2 < m_1$ respectively, move only under the influence of the force of gravitational interaction in circular orbits around a common center of mass C. The distance between the bodies remains unchanged and equal to R_0.

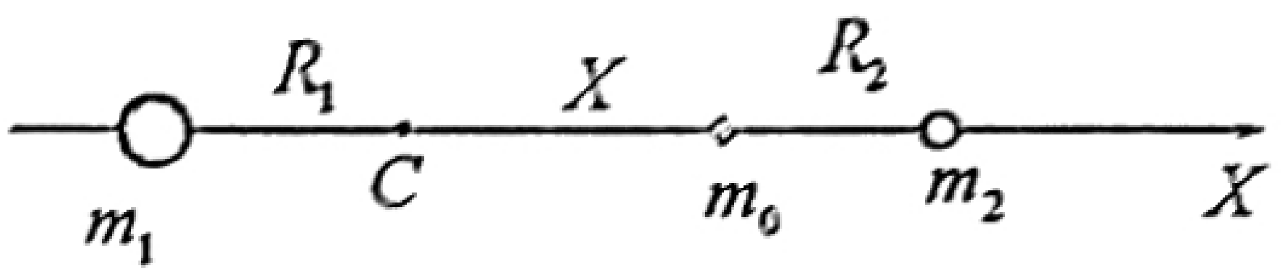


2.1 Find the radii R_1 and R_2 of the trajectories of the bodies. Express your answer through the quantities R_0, m_1, m_2.

2.2 Find the angular velocity ω_0 of the bodies rotations. Express your answer through the quantities R_0, m_1, m_2 and the gravitational constant G.

Lagrange points in a three-body system

Lagrange points, libration points (Latin librātiō – swinging) or L-points are points in the space of a system of two massive bodies in which a third body with negligible mass, not affected by any other forces other than gravitational ones from the first two, can remain motionless relative to these bodies.

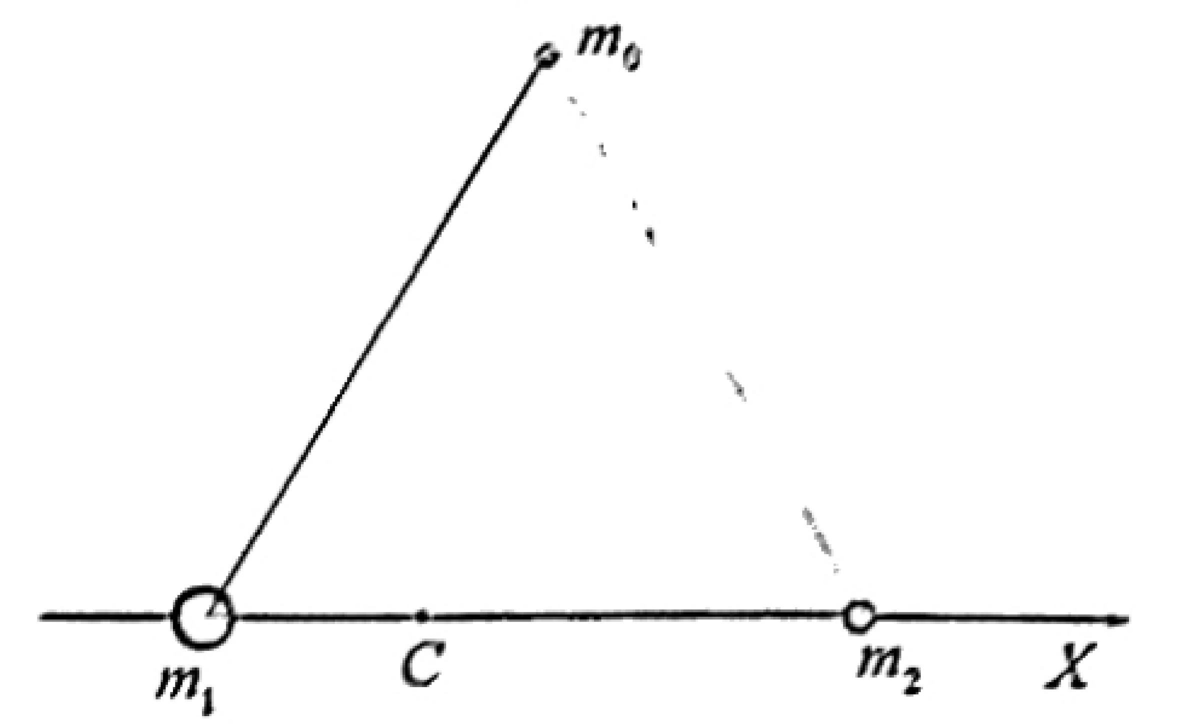


Consider adding a third body to the system described above, the mass of which m_0 is significantly less than the masses of the first two bodies, $m_0 \ll m_1, m_2$. Introduce the X axis along a straight line passing through the bodies m_1, m_2, as shown in the figure. The origin is compatible with the center of mass C. To simplify mathematical transformations, use the following quantities:

- as a unit of length, use the distance R_0 between bodies m_1, m_2; the position of a small body is determined by the dimensionless coordinate $x = X/R_0$, which can vary from minus to plus infinity;
- as a unit of force use the quantity $F_0 = m_0\omega_0^2 R_0$;
- introduce the dimensionless parameter $\mu = \frac{m_2}{m_1+m_2}$.

2.3 Obtain an exact equation for determining the x coordinates of Lagrange points lying on the X axis. In addition to the desired x coordinate, this equation should include only the parameter μ.

2.4 Draw a schematic graph of the projection onto the X axis of the force F_x acting on a small body from two massive bodies at $\mu = 0.20$. Express your answer in terms of relative units $f_x = F_x/F_0$ and $x = X/R_0$.



2.5 Determine the possible number of Lagrange points that exist on the X axis.

2.6 Calculate the numerical values of the coordinates of these points with an error not exceeding $\Delta x = 0.05$ if $\mu = 0.20$.

2.7 Prove that the point lying at the vertex of a regular triangle constructed on the segment m_1, m_2 is a Lagrange point.

Lagrange points in the Solar System

For calculations, use the following numerical values of astronomical quantities: solar mass $M_1 = 1.99 \cdot 10^{30}$ kg; Earth mass $M_2 = 5.97 \cdot 10^{24}$ kg; the Earth's orbit can be considered a circle with radius $R_0 = 1.50 \cdot 10^8$; Jupiter's orbital period around the Sun is 11.9 Earth years.

SOHO is a solar observing spacecraft, a joint project of ESA and NASA, which was launched on December 2, 1995 and began operations in May 1996. The device is located on the straight line connecting the Sun and the Earth, its position relative to the Earth remains practically unchanged over time.

2.8 Determine at what distance l_S from the Earth the SOHO spacecraft is located. Express the formula in terms of the ratio of the masses of the Earth and the Sun and the radius of the Earth's orbit. Calculate the numerical value of this distance.

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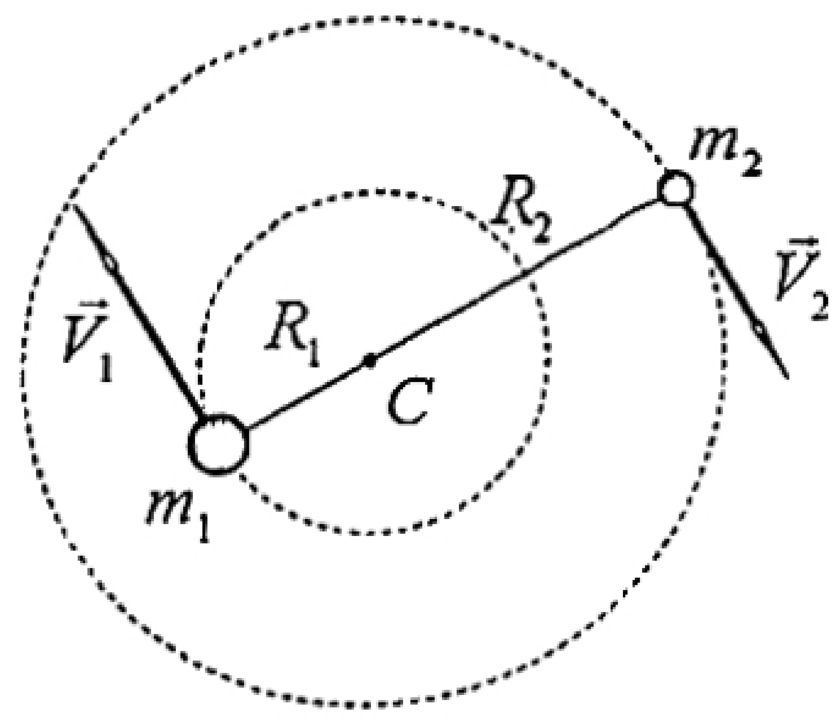

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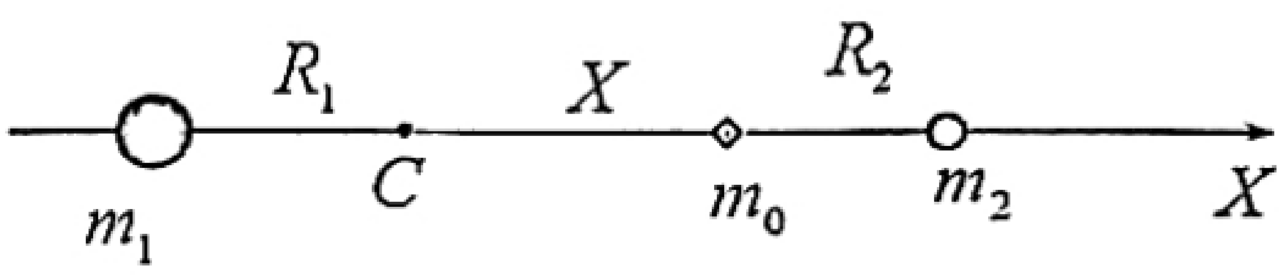

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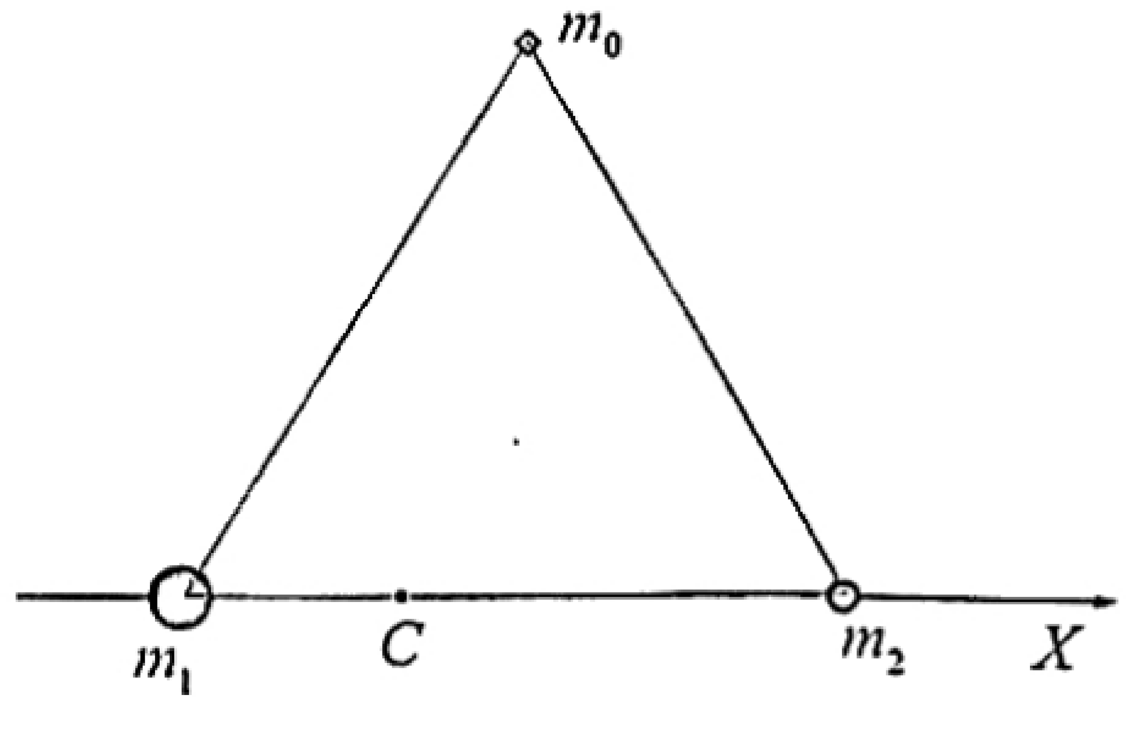

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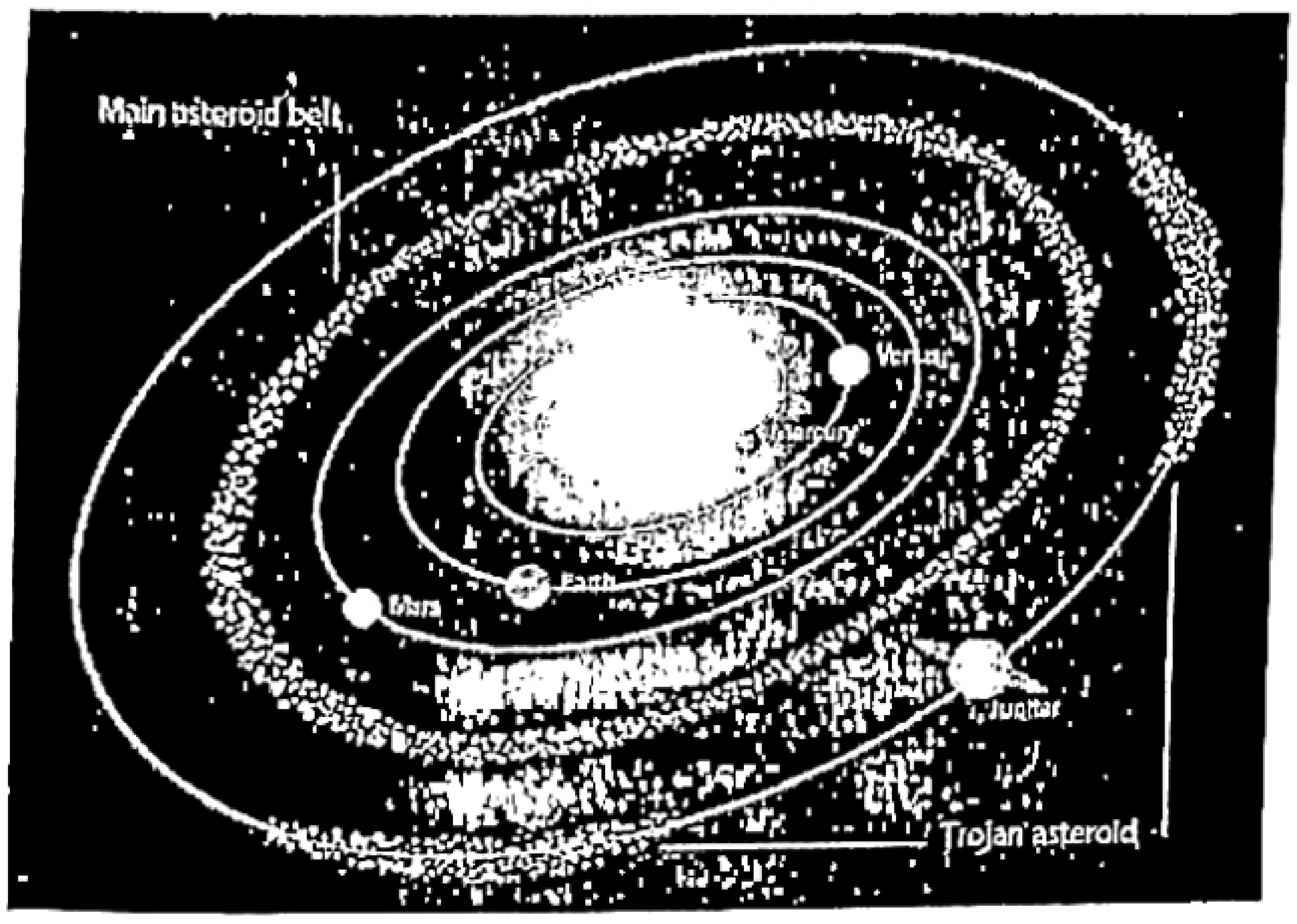
On December 25, 2021, a spacecraft equipped with the James Webb infrared telescope was successfully launched from the Kourou spaceport using an Ariane 5 rocket. In January 2022, this device reached its permanent deployment point. The position of the telescope is to remain virtually unchanged relative to the Earth, and it is assumed to stay in the Earth shadow all the time.

2.9 Determine at what distance l_W from the Earth the James Webb telescope is located. Express your answer in terms of the Earth-to-Sun mass ratio and the radius of the Earth's orbit. Calculate the numerical value of this distance.



Small planets, called asteroids, are distributed extremely unevenly in the Solar System: in addition to the well-known asteroid belt, located between the orbits of Mars and Jupiter, there are two large groups of asteroids moving near the orbit of Jupiter. These groups of asteroids are called "Trojan" asteroids.

2.10 Calculate the distances l_J from Jupiter to the centers of groups of Trojan asteroids.

Problem 3. Geometric optics and photodetector (10.0 points)

In this problem, the laws of geometric optics are to be studied using a photodetector, which is a device for converting light energy into electrical energy. It is assumed that the absorbing surface of the photodetector is flat and when it is connected to a milliammeter, it shows a current value proportional to the power of the absorbed electromagnetic radiation.

Geometric optics is a branch of optics that studies the laws of light propagation in transparent media, the reflection of light from mirror-reflecting surfaces, and the principles of constructing images when light passes through optical systems without taking into account its wave properties. The basic concept of geometric optics is a light beam. This implies that the direction of the flow of radiant energy (path of the light beam) does not depend on the transverse dimensions of the light beam. Let us recall the laws of geometric optics necessary for this problem solution.

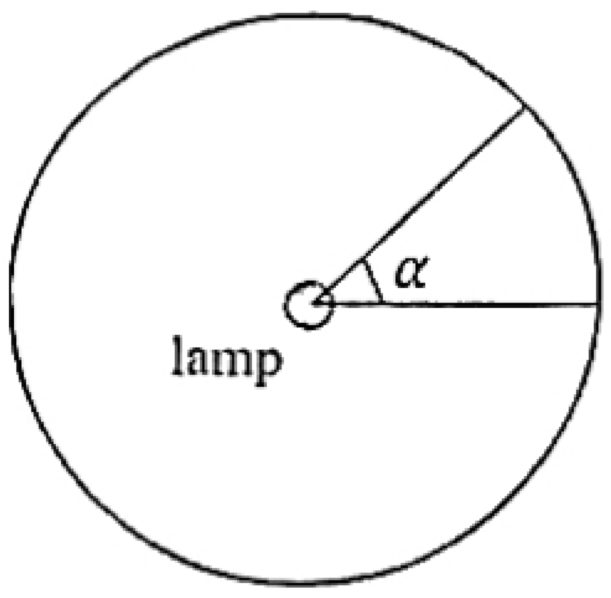
3.1 When reflected from a flat surface, the incident and reflected rays lie in the same plane with the normal at the point of incidence. Let the incident ray make an angle α with the normal, and the reflected ray make an angle β. Write down the relationship between these angles.

3.2 When light falls from air onto a flat surface of a material with refractive index n, the incident and refracted rays also lie in the same plane with the normal at the point of incidence. Let the incident ray make an angle α with the normal, and the refracted ray make an angle β. Write down the relationship between these angles.

In the following parts of this problem, we consider only situations in which, in the processes of light refraction, the above-mentioned angles are small and the approximation of paraxial optics holds.

3.3 Rewrite the answer to 3.2 in the case of small angles of incidence and refraction.

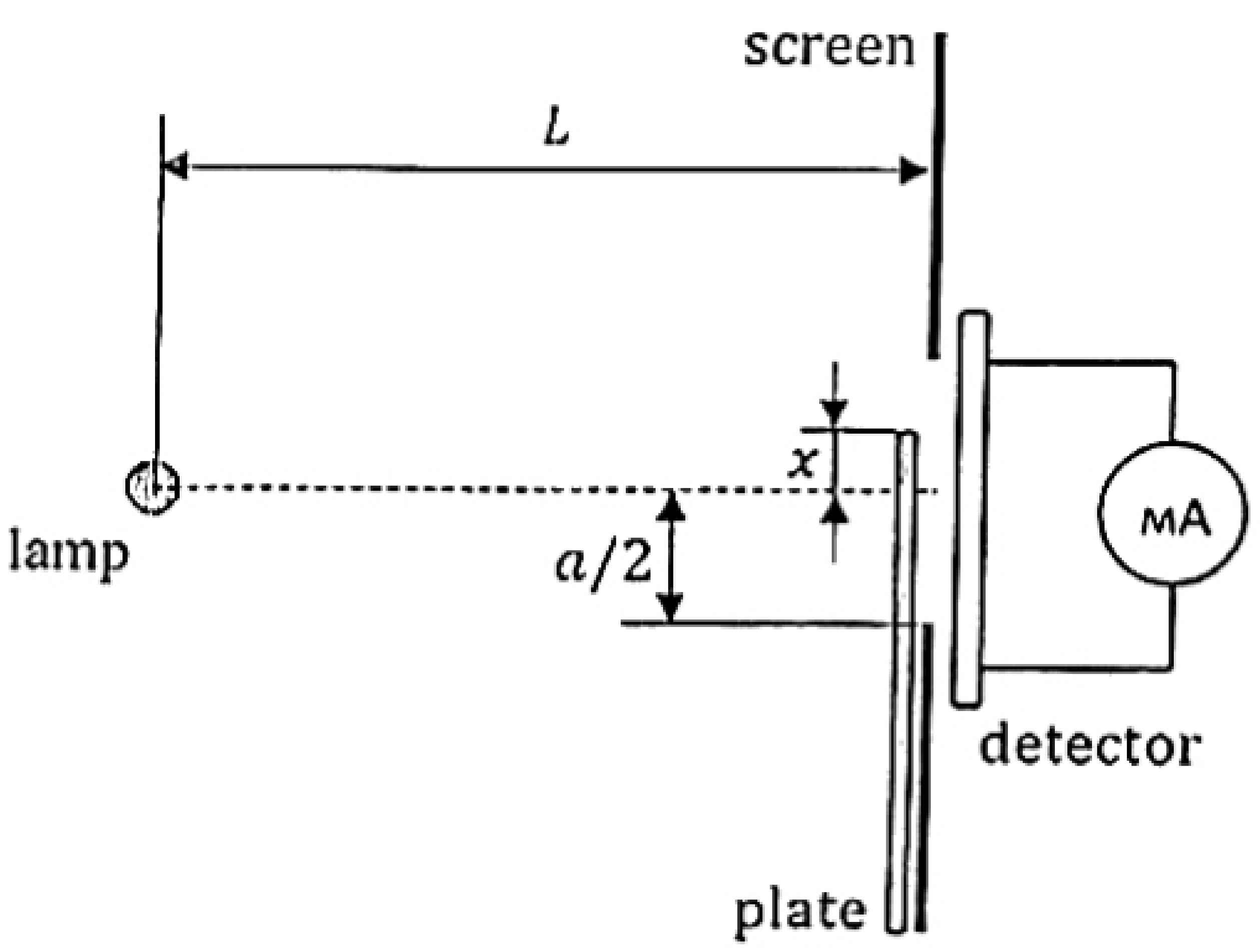
In this problem, the radiation source is assumed to be a very long, thin cylindrical lamp, the surface of which emits equally in all directions.



3.4 Let the radiation power of the lamp from its entire surface be W. Determine the radiation power W_α concentrated in a plane angle α as shown in the figure.

Thin plate

Let the light source described above illuminate an opaque screen in which a rectangular slit $a = 5.00$ cm wide is made. The lamp is located parallel to the edges of the slit directly opposite its center at a distance of $L = 100$ cm, and immediately behind the slit the surface of the photodetector is placed in such a way that it captures the entire luminous flux passing through the slit. The photodetector itself is connected to a milliammeter. A thin rectangular plate can move in parallel along the screen, the upper edge of which is parallel to the edges of the slit, while the material of the plate is translucent with a transmittance coefficient of $\tau = 0{,}500$. In the absence of the plate, the milliammeter shows the current value $I_0 = 10.0$ mA.



3.5 Let now x be the coordinate from the center of the slit to the edge of the plate with the axis vertically directed upward. Plot a graph of the deviation of the milliammeter readings $\Delta I(x)$ in milliammeteres from the initial reading I_0 from the x coordinate in the range of -5.00 cm to 5.00 cm. Explicitly indicate the coordinates of all characteristic points on the graph.

Thick plate

Let us leave the observation scheme the same, but replace the thin plate with a thick plate of thickness d, while its horizontal surface is a perfectly reflective mirror on both sides, and the refractive index of the material is n. For this part, consider the numerical values of the parameters a, L, τ and I_0 unknown. For clarity, the layout diagram is shown in Figure 1, and Figure 2 shows the corresponding measurement results in the range of changes in the x coordinate from -3.00 cm to 0.00 cm and three characteristic points x_1, x_2, x_3 of the graph are marked. Here $\Delta I(x)$ is the deviation of the milliammeter readings in milliammeteres from the initial reading I_0 in the absence of the plate.

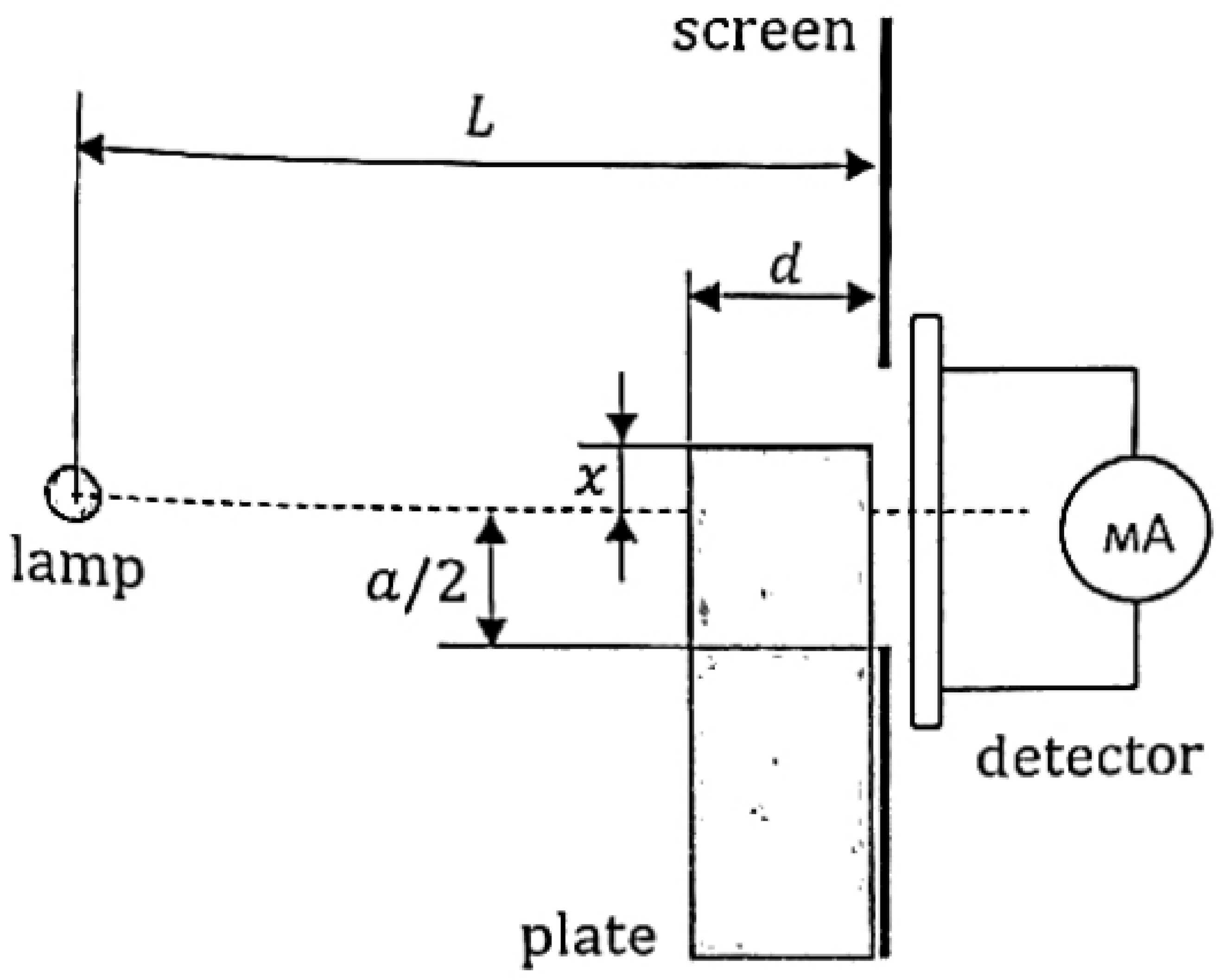


Figure 1. Scheme of measurements.

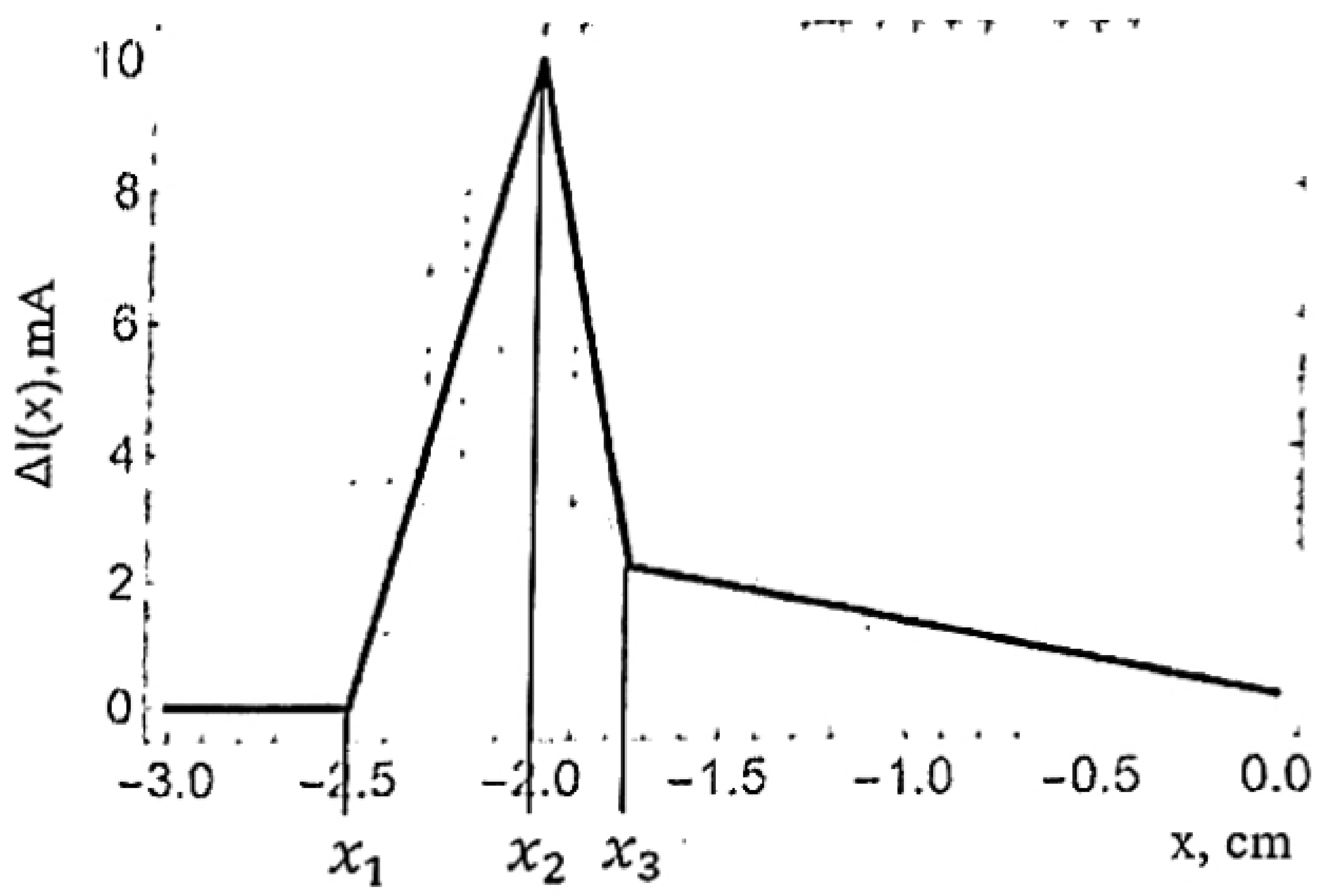


Figure 2. Measurement results.

3.6 Express the coordinate values of point x_1 in terms of the values a, L, d, n, τ.

3.7 Express the coordinate values of point x_2 in terms of the values a, L, d, n, τ.

3.8 Express the value of ΔI_{max} at point x_2 in terms of the quantities a, L, d, n, τ and I_0.

3.9 Express the coordinate values of point x_3 in terms of the values a, L, d, n, τ.

3.10 Find the slope coefficient $d\Delta I(x)/dx$ of the straight line in the area to the right of point x_3 and express it through the quantities a, L, d, n, τ, I_0.

3.11 Based on the graph and parameters obtained in 3.6-3.10, determine the numerical values of the parameters a, n, τ, I_0, as well as the ratio L/d.

3.12 Based on the obtained numerical data, construct the right side of the graph as the x coordinate changes from 0.00 cm to 3.00 cm, indicating the coordinates of all characteristic points on the graph.

Mathematical hint for the theoretical competition

The following formulas may be useful:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$ is a fixed number, C refers to an arbitrary constant;

$\int \frac{dx}{x} = \ln|x| + C$, where C stands for an arbitrary constant;

$(1+x)^\gamma \approx 1 + \gamma x + \frac{\gamma(\gamma-1)}{2}x^2$, for $|x| \ll 1$ and any value of γ;

$\tan x \approx \sin x \approx x$, for $|x| \ll 1$;

$\ln(1+x) \approx x$, for $|x| \ll 1$.