THEORETICAL COMPETITION

February 2, 2023

Please read this first:

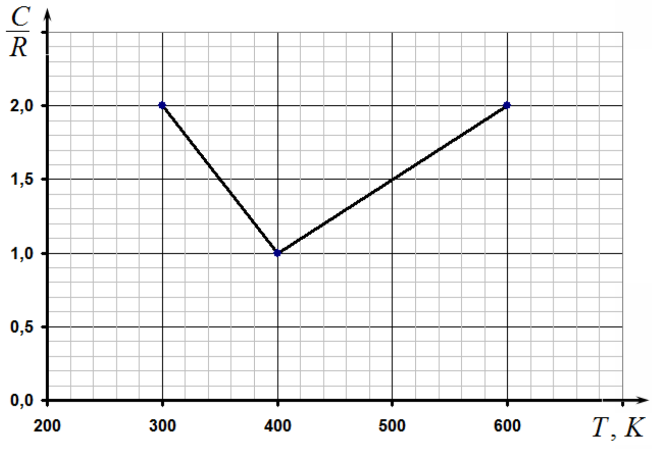
1. The duration of the theoretical competition is 4 hours. There are three problems.
2. You can use your own calculator for numerical calculations.
3. You are provided with ***Writing sheet*** and additional white sheets of paper. You can use the additional sheets of paper for drafts of your solutions, but these sheets will not be graded. Your final solutions should be written on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures, and plots.
4. Use only the front side of ***Writing sheets*.** Write only inside the boxed areas.
5. Start putting down your solution to each problem on a new ***Writing sheet***.
6. Fill in the boxes at the top of each ***Writing sheet*** with your country (**Country),** your student code (**Student Code**), problem number (**Question Number)**, the progressive number of each ***Writing sheet*** **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be graded, put a large X across the entire sheet and do not include it in your numbering.

Problem 1 (10.0 points)

This problem consists of three independent parts.

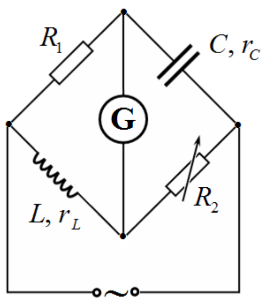
Problem 1.1 (4.0 points)

One mole of an ideal monatomic gas, located in the cylinder under the piston, is quasi-statically heated from the temperature $T_1 = 300\ \text{K}$ to $T_2 = 600\ \text{K}$, changing its volume in such a way that the dependence of the heat capacity C of the gas on its temperature has the form shown in the figure below. Find the work done on the gas from the state in which the graph of its volume versus temperature reaches a local maximum, to the state in which the same graph reaches a local minimum. Express your answer in joules assuming that $R = 8.31\ \text{J/K}$ is the universal gas constant.



Problem 1.2 (3.0 points)

The figure on the right shows an AC bridge circuit. The resistance $R_1 = 2.5\ \text{k}\Omega$, the inductance $L = 1\ \text{H}$, the resistance of the inductance $r_L = 1\ \Omega$ are all known. At the frequency of the alternating sinusoidal voltage $\nu = 100\ \text{Hz}$, the balance of the bridge occurs at $R_2 = 800\ \Omega$. It turnes out that when the frequency of the alternating current is doubled, the balance of the bridge is not violated. Find the leakage resistance r_C of the capacitor and its capacitance C .



Problem 1.3 (4.0 points)

The Nobel Prize in Physics for 2019 was awarded to the Swiss astronomers M. Major and D. Quelo for the discovery of non-luminous satellite planets (exoplanets) around stars. Consider the planet Jupiter in our solar system as an exoplanet for the Sun. The orbital period of Jupiter is $T_J = 11.9\ \text{year}$ at a mass of $M_J = 1.90 \cdot 10^{27}\ \text{kg}$, while the mass of the Sun is equal to $M_S = 1.99 \cdot 10^{30}\ \text{kg}$. Consider also known the speed of the Earth on its orbit $v_E = 29.8\ \text{km/s}$, and assume that the orbits of all planets are circular. Let an observation be carried out by a spectrometer located far away from outside the solar system so that the observer is in the plane of motion of the Sun–Jupiter system. Find the minimum resolution of the spectrometer $R_{\min}$, which allows one to positively detect the presence of the massive exoplanet Jupiter near the Sun. Consider the speed of light to be $c = 2.99 \cdot 10^8\ \text{m/s}$.

Note: The resolution of a spectrometer is its ability to distinguish between two closely spaced spectral lines, which is characterized by a dimensionless parameter $R = \lambda / \Delta\lambda$, where $\Delta\lambda$ denotes the smallest difference between the wavelengths of two lines still recorded as separate by the instrument, and λ stands for the average wavelength of those two resolvable lines.

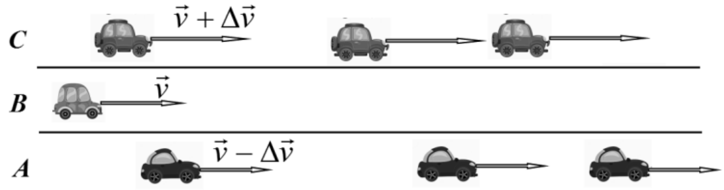
Problem 2. Fermi acceleration (10.0 points)

Cosmic rays contain extremely high energy particles. A possible mechanism for the appearance of such particles is called Fermi acceleration.

Fermi acceleration is a stochastical mechanism for the acceleration that charged particles experience when they are repeatedly reflected, usually by magnetic mirrors. In this problem, we consider the main ideas underlying this paradoxical, at first glance, phenomenon.

Why are there more oncoming cars than overtaking cars?

On a motorway with three lanes ***A, B, C,*** cars move at a constant speed in one direction: along the central lane ***B***, a car moves at a speed of $v = 90\ \text{km/h}$; cars move along lane ***A*** at speeds $v - \Delta v = 80\ \text{km/h}$; cars move along lane ***C*** at speeds $v + \Delta v == 100\ \text{km/h}$. The distance between cars is the same on each of lanes ***A*** and C, the number of cars per unit of lane length for each of them is $n = 5.0\,\text{km}^{-1}$.

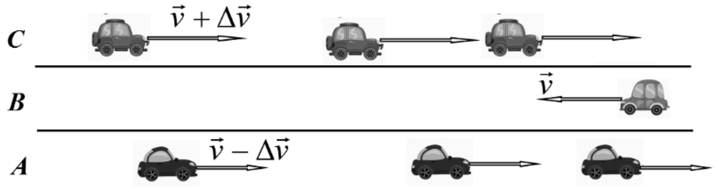


Consider the car moving in lane ***B***.

2.1 Calculate how many cars N_1 in lane ***A*** overtake the car ***B*** within the time period of $t = 1.0\ \text{min}$, as well as the time τ_1 between two consecutive overtakes.

2.2 Calculate how many cars N_2 in lane ***C*** overtake the car in lane ***B*** within the time period of $t = 1.0\ \text{min}$, as well as the time τ_2 between two consecutive overtakes.

Now let the car move in lane ***B*** towards the cars moving in lanes ***A*** and ***C***. The speeds of all cars and their density on each lane remain the same.

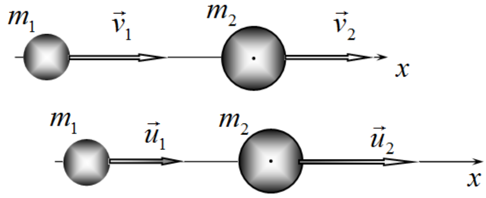


2.3 Calculate the number of cars N_3 in lane ***A*** and N_4 in lane ***C*** that the car ***B*** encounters within the time period of $t = 1.0\ \text{min}$, as well as the corresponding times τ_3 and τ_4 between two consecutive encounters.

Elastic collision

In this part, we consider the classical problem of elastic collision of two bodies. The main purpose of this consideration is to determine the conditions under which the kinetic energy of one of the selected bodies increases as a result of the collision.

Two elastic balls, whose masses are equal to m_1 and m_2, respectively, move along the axis x. The speed of the first ball before the collision is v_1, whereas the speed of the second is v_2. Let us denote the speeds of the balls after an absolutely elastic central collision as u_1 and u_2, respectively. The speeds of the balls should be understood as the projections of their velocities on the x axis, therefore, they can be both positive and/or negative.



2.4 Express the velocities of the balls u_1 and u_2 after the collision in terms of their velocities before the collision v_1 and v_2, as well as their masses.

Let us denote the ratio of the ball masses as $\mu = \frac{m_2}{m_1}$, the ratio of the speeds of the first ball after and before the collision as $\eta_1 = \frac{u_1}{v_1}$ and the ratio of the velocities of the balls before the collision as $\eta_2 = \frac{v_2}{v_1}$. For definiteness, consider that $v_1 > 0$.

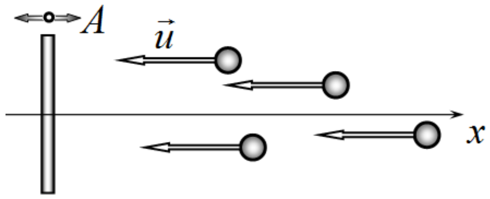
2.5 Draw a set of graphs that represents the dependence of the parameter η_1 on the parameter η_2 for all characteristic values of the ball mass ratios μ.

2.6 Find the relation between the parameters η_2 and μ, at which the first ball increases its energy as a result of the collision.

2.7 Consider the case of a light ball colliding with a heavy one such that $m_2 >> m_1$. In this limiting case, find the speed of the first ball $\tilde{u}_1$ after the collision and determine the range of velocities η_2 of the heavy ball before the collision, at which the energy of the light ball increases as a result of the collision.

The simplest Fermi acceleration model

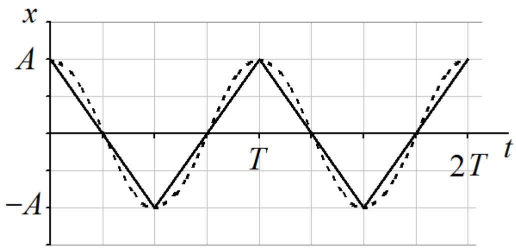
A massive plate, located perpendicular to the x axis, performs harmonic oscillations in the direction of the x axis. The oscillation amplitude is A with the period being T. Light balls move with equal speeds u in the direction of the plate along the x axis such that the times of ball arrivals to the still plate are randomly and uniformly distributed.



2.8 Express the maximum speed of the plate V_0 in terms of the amplitude and period of its oscillations.

2.9 Calculate the fraction φ of incident balls that are to increase their kinetic energy after the collision. Express your answer in terms of u and V_0. For a numerical estimate, consider the following two cases separately: a) $u = 1.5V_0$; b) $u = 0.50V_0$.

Let us approximate the harmonic law of the plate motion by a piecewise linear function, see the figure below, i.e. assume that the modulus of the plate speed remains constant at the same values of the amplitude and period of oscillations.



2.10 Express the value of the plate speed modulus V in terms of the amplitude and period of its oscillations.

2.11 Calculate how many times $\varepsilon = \frac{E}{E_0}$ the average energy of the incident balls changes, where E_0 is the kinetic energy of the balls before the collision, and E symbolizes the average energy of the balls after the collision with the oscillating plate. For a numerical estimate, consider the following two cases separately: a) $u = 1.5V$; b) $u = 0.50V$.

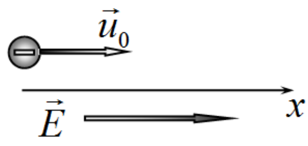
Problem 3. Magnetron (10.0 points)

Both electric and magnetic fields act on moving charged particles, which in a certain way change the character of their motion. Everywhere below it is assumed that charged particles move in a vacuum, and the radiation of electromagnetic waves can be neglected.

In this problem, the motion of electrons, which are classical point-like particles, is considered. When carrying out numerical calculations, consider, where necessary, that an electron has a negative electric charge, whose modulus is equal to $e = 1.60\cdot 10^{-19}\,\text{Cl}$, and its mass is $m = 9.11\cdot 10^{-31}\,\text{kg}$. The electric and magnetic constants are equal $\varepsilon_0 = 8.85\cdot 10^{-12}\,\text{F/m}$ and $\mu_0 = 1.26\cdot 10^{-6}\,\text{H/m}$, respectively, the Boltzmann constant is denoted as $k_B = 1.38\cdot 10^{-23}\,\text{J/K}$.

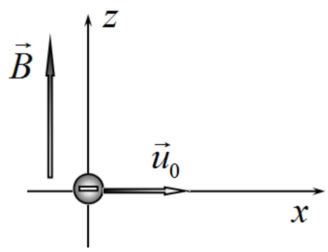
Electron motion in electric and magnetic fields

At the initial moment of time $t = 0$ an electron is at rest at the origin of coordinates, and a uniform electric field with the strength E is applied along the positive direction of the x axis. The electron is given an initial velocity u_0, also directed in the positive direction of the x axis.

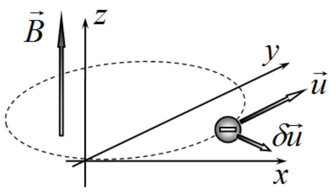


3.1 Determine the maximum value of the electron coordinate $x_{\max}$ for the entire time of its motion.

Now, at the initial moment of time $t = 0$, an electron is at rest at the origin of coordinates, and a uniform magnetic field with the induction B is applied along the positive direction of the z axis. The electron is given an initial velocity u_0 directed in the positive direction of the x axis.

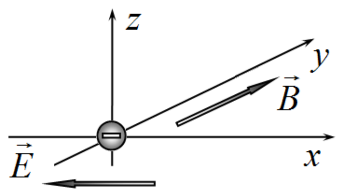


3.2 Determine the maximum value of the electron coordinate $x_{\max}$ for the entire time of its motion.

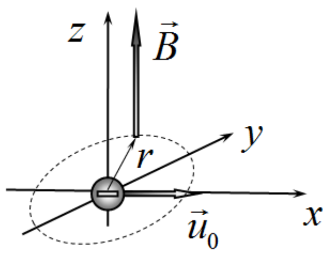


3.3 In the process of moving along the trajectory, at some moment of time, an additional velocity δu is given to the electron in the direction perpendicular to its current velocity u and lying in the plane of its initial trajectory, such that $\delta u \ll u$. Determine the period of the arisen two-dimensional oscillations of the electron with respect to its initial unperturbed trajectory.

Again, at the initial moment of time $t = 0$ an electron is at rest at the origin of coordinates, and a uniform electric field with the strength E is applied along the negative direction of the x axis. In addition, a uniform magnetic field with induction B is created along the positive direction of the y axis. The electron is released with the zero initial velocity.



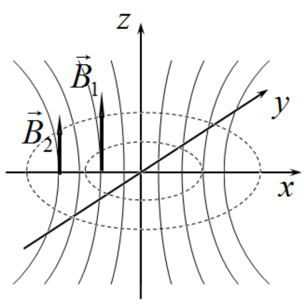
3.4 Determine the maximum value of the electron coordinate $x_{\max}$ for the entire time of its motion.



At the initial moment of time $t = 0$ an electron is at rest at the origin of coordinates, and a magnetic field is applied along the positive direction of the z axis, such that the inductionшт in the plane xy depends on the distance r to the z axis according to the law $B = \alpha r = \alpha(x^2 + y^2)^{1/2}$. The electron is given an initial velocity directed in the positive direction of the x axis.

3.5 Determine the maximum distance $r_{\max}$ from the electron to the z axis for the entire time of its motion.

In some region of space, an axisymmetric magnetic field is created with respect to the z axis. At the moment of time $t = 0$, the magnetic field is absent, and then it begins to slowly increase, such that at all points in the xy plane the field induction vector is directed along the z axis. After a certain period of time, the induction of the magnetic field reaches its final value and ceases to change. The final field distribution in the xy plane has the form



$$B(r) = \begin{cases} B_1, & 0 < r \le r_1 \\ B_2, & r_1 < r \le r_2 \\ 0, & r > r_2 \end{cases},$$

where r denotes the distance to the z axis, r_1 and r_2 are known quantities.

At a moment in time $t = 0$, the electron is at rest at some point lying in the xy plane. When the field is turned on, the electron begins to move along a circle in the xy plane, whose center lies on the z axis.

3.6 Determine under what conditions for the ratio B_1 / B_2 the described situation turns possible.

Cylindrical magnetron

Magnetron is an electronic electrovacuum device, in which the amount of flowing current is controlled by electric and magnetic fields. Let us consider the simplest magnetron, consisting of a conducting coaxial long cylindrical cathode and anode with radii $a = 3.00 \cdot 10^{-1}\,\text{mm}$ and $b = 6.00\ \text{mm}$, respectively, located inside a vacuum tube. The lamp is placed in the center of a cylindrical solenoid, whose axis coincides with the axes of the cathode and anode, and the number of turns is $N = 2590$ with the length being equal $L = 210\ \text{mm}$ and the diameter of $D = 105\ \text{mm}$. The potential difference between the cathode and the anode is constant and equal to $V_0 = 75.0\ \text{V}$. Consider that the electrons leaving the cathode are formed due to thermionic emission and have a zero initial velocity, neglect the space charge of the electron cloud in the lamp.

3.7 Calculate the potential difference V between the cathode and a point in space located at a distance $r = 3.00$ мм from the common axis of the solenoid and the lamp.

3.8 Calculate the smallest solenoid current $I_{\min}$ at which the current in the magnetron between the cathode and anode vanishes completely.

3.9 Find the condition for the cathode temperature T, under which the initial electron velocity can indeed be considered zero.

Mathematical hints for the theoretical problems

The following formulas may be useful:

$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, where $n \ne -1$ is a fixed number, C refers to an arbitrary constant;

$\int \dfrac{dx}{x} = \ln|x| + C$, where C stands for an arbitrary constant;

$\int \dfrac{dx}{(x^2+a^2)^{3/2}} = \dfrac{x}{a^2(x^2+a^2)^{3/2}} + C$, where a, C stand for arbitrary constants;

$(1+x)^\gamma \approx 1 + \gamma x + \dfrac{\gamma(\gamma-1)}{2}x^2$, for $|x| \ll 1$ and arbitrary γ;

$\ln(1+x) \approx x$, for $|x| \ll 1$.