Janssen effect

The Janssen effect is a characteristic feature of granular materials. When granules are poured into a container, a network of contacts between particles leads to the formation of force chains (see Fig.) that redistribute the stress of the granular column towards the side walls.

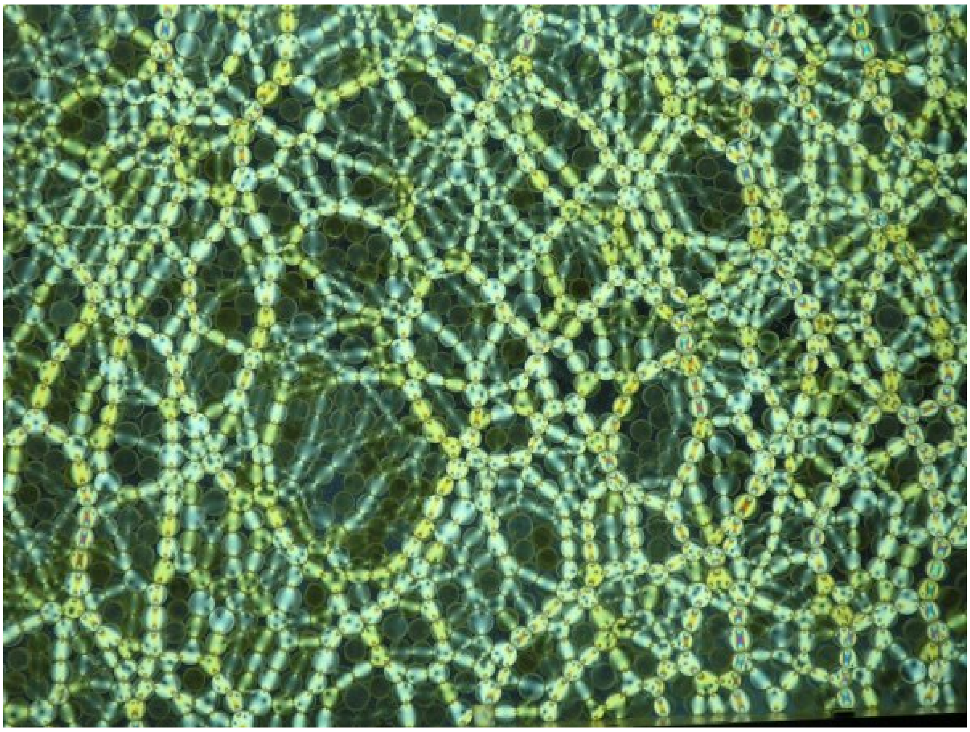

Fig. 1: Force chains

This ability of granular materials to self-organize stress distribution through force chains distinguishes them from conventional liquids and solids, making their study important for various fields of science and engineering. This is particularly important in industry, where large quantities of raw materials are typically stored as granules in silos (storage facilities for granular material).

This problem is devoted to investigating this effect by examining the behavior of spherical granules in vessels of various shapes.

Throughout the problem, disregard the contribution of atmospheric pressure. All pressures refer to gauge pressure relative to atmospheric pressure.

The following integrals may be useful:

$$\int \frac{dx}{a+x} = \ln|a+x| + C$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$\int e^x dx = e^x + C$$

Part A. Hydrostatics (0.5 points)

A cylindrical vessel with vertical walls and a bottom area S (see Fig.) is completely filled with water of total mass m. The density of the water is ρ.

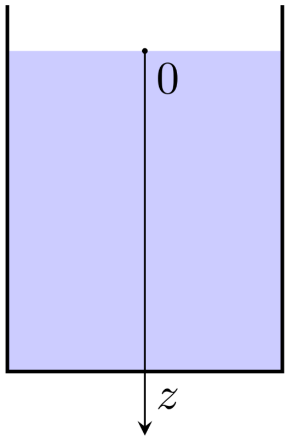

Fig. 2: Cylindrical vessel with water

A.1	Determine the pressure p_1 at the bottom of the vessel. Express your answer in terms of m, g, and S.	0.2pt

A.2	Determine the pressure $p(z)$ on the vessel walls as a function of depth z. Express the answer in terms of ρ, g, and z.	0.3pt

Part B. Janssen effect in a cylindrical vessel with vertical walls (2.9 points)

This part investigates how the pressure exerted by the granules on the bottom of a vertical-walled cylindrical vessel depends on the mass of the granules.

For parts **B**, **C** and **D** of the problem, work within the following model:

- ρ is the **bulk** density of the material (average mass of granules per unit volume of the vessel).
- Before the experiments, the vessel is shaken, ensuring a completely random arrangement of the granules. There is a very large number of granules, and the free surface is horizontal.
- z is the depth measured from the free surface ($z = 0$ at the averaged granule-air interface, with the z-axis directed vertically downward).
- Assume that the granular pressure against the walls is directly proportional to the vertical pressure in the vessel: $\sigma(z) = KP(z)$. Here, K is the constant coefficient. Throughout the remainder of the problem, understand the pressure of the granules as the averaged normal force acting per unit surface area. For example, for the pressure on the vessel walls:

$$\sigma(z) = \left\langle \frac{dF_{\text{hor}}}{dS}(z) \right\rangle .$$

- Assume this quantity is independent of the radial direction. Also, assume the vertical pressure is independent of the radial direction and the distance from the vessel axis.

- μ is the friction coefficient of the granular material against the vessel wall. Assume that the friction force acting on the granules is vertical and maximal in magnitude. Neglect the phenomenon of stagnation; that is, assume that the maximum possible value of the friction force is $dF = \mu\sigma dS$. In parts **B** and **C**, assume that friction minimizes the pressure on the bottom of the vessel, i.e., the friction forces acting on the granules are directed vertically upwards.
- R is the inner radius of the cylindrical vessel.
- $g = 9.8\ m/s^2$ is the acceleration due to gravity.

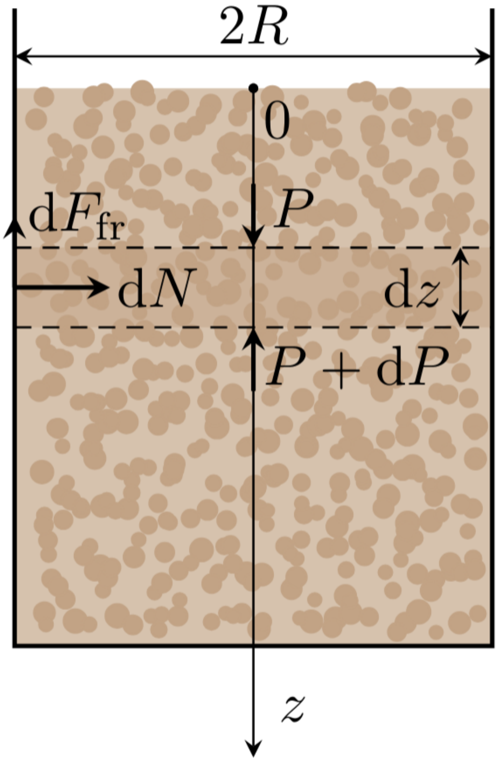

Fig. 3: Cylindrical vessel with granules

Consider a thin layer of granular material at depth z with thickness dz. The radius of this layer is equal to the inner radius of the vessel.

B.1	Write down the equilibrium condition for this layer. The equation may include the pressures $P(z)$ and $P(z+dz)$, dz, the average pressure on the vessel walls $\sigma(z)$, the mass of the layer dm, as well as μ, g, and R.	0.5pt

B.2	Show that the bottom pressure P as a function of the fill height h of the granular column can be written as: $$P = A + Be^{-h/\lambda}.$$ Determine the parameters A, B, and λ. Express your answers in terms of the bulk density ρ, as well as g, R, K, and μ.	1.2pt

The graph below shows the dependence of the granular pressure **on the bottom** on the granules level in the vessel, $P_0(H)$.

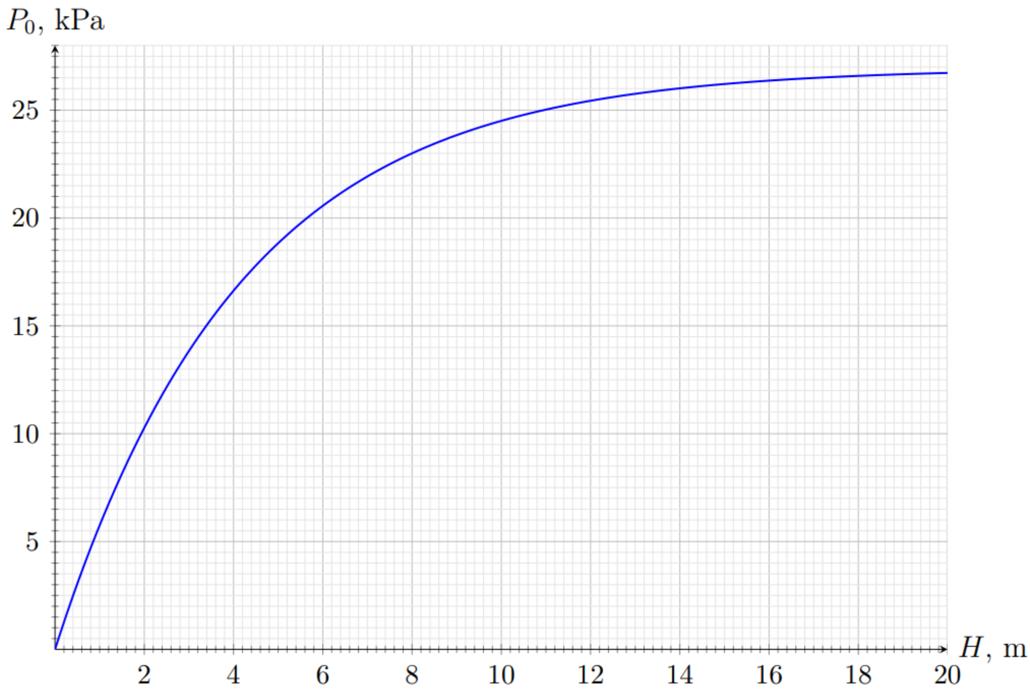

Fig. 4: Graph of the $P(h)$ dependence

B.3	Determine the bulk density of the granules ρ and the coefficient K. Give the answers to two significant figures. For this part, assume the numerical value of the vessel radius is $R = 50\ \mathrm{cm}$ and the coefficient of friction is $\mu = 0.30$.	1.2pt

Part C. Conical vessel (2.3 points)

In this part of the problem, we investigate the pressure distribution of granules in a conical vessel, oriented with the bottom facing upward and **completely** filled with granules. This calculation is crucial, for example, for designing grain storage silos (see the figure below).

Fig. 5: Conical silo

Assume the base radius R and the cone's half-angle $\alpha \ll 1$ are known.

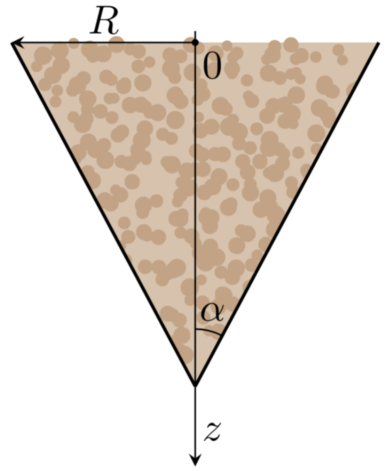

Fig. 6: Granules in a conical vessel

C.1	In the specified approximation, obtain the dependence of the vessel's cross-sectional radius on height $r(z)$. Express the answer in terms of R, α, and z.	0.1pt

Let us again consider a thin layer of granular material analogous to the layer in **B1**.

C.2	Write down the equilibrium condition for the layer. The equation may include the pressures $P(z)$ and $P(z+dz)$, dz, the average pressure on the vessel walls $\sigma(z)$, the mass of the layer dm, as well as μ, g, $r(z)$, $r(z+dz)$, α, and R.	0.6pt

For the remainder of this part, assume that $\alpha \ll \mu$. Note that $K\mu$ and α may be of the same order of magnitude. Let us introduce the notation:

$$z_{\text{max}} = \frac{R}{\alpha}.$$

Assume henceforth that the dependence of the vertical pressure P inside the vessel on the coordinate z is given by:

$$P(z) = A\left(1 - \frac{z}{z_{\text{max}}} - \gamma\left(1 - \frac{z}{z_{\text{max}}}\right)^{\beta}\right),$$

where A, γ, and β are constants. The system parameters are chosen such that $A, \beta > 0$.

C.3	Determine the parameters A, γ, and β. Express the answer in terms of the bulk density of the granules ρ, g, z_{max}, K, μ, and α.	0.8pt

C.4	Sketch a qualitative graph of $P(z)$ and determine the ratio of the maximum vertical pressure P_{max} to $\rho g z_{\text{max}}$ for $\alpha = 0.2K\mu$. Provide the answer to two significant figures.	0.8pt

Part D. What if the vessel is lifted? (1.5 points)

In this part of the problem, we again consider a vertical cylindrical massless vessel of radius R.

A layer of granular material of height H is poured into it. A rod of radius a is vertically inserted into the center of the container, touching the bottom. The bulk density of the granules is ρ, the coefficient of friction of the material against the container wall is μ_1, and the coefficient of friction of the material against the rod is μ_2. The rod begins to be slowly pulled upward at a constant low speed. Due to friction, the vessel with the granules begins to rise slowly. Throughout this motion, the granules remain stationary **relative to the vessel**.

The coefficient K is still constant both for the interaction of granules with the vessel walls and for the interaction of granules with the rod.

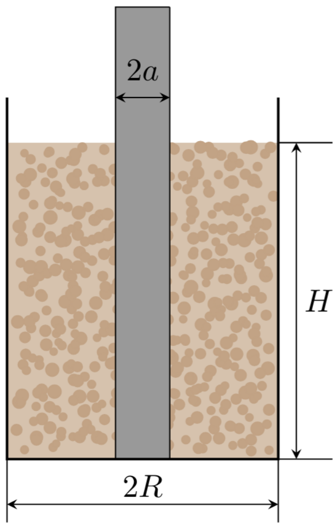

Fig. 7: Rod in a vessel with granules

Let us again consider a thin layer of granular material analogous to the layer in **B1**. Now, this layer is in contact with both the vessel and the rod.

D.1	Write down the equilibrium condition for the layer. The equation may include the pressures $P(z)$ and $P(z+dz)$, dz, the average pressure on the rod and vessel walls $\sigma(z)$, the mass of the layer dm, as well as μ_1, μ_2, g, a, and R.	0.4pt

D.2	Show that the bottom pressure P as a function of the fill height h of the granular column can be written as: $$P = A + Be^{-h/\lambda}.$$ Determine the parameters A, B, and λ. Express the answers in terms of the bulk density of the granules ρ, as well as g, a, R, μ_1, μ_2, and K.	0.3pt

D.3	Determine the minimum level H of the granules relative to the bottom at which the cylinder can be lifted as described above. Assume that $H \ll \lambda$. Express the answer in terms of a, R, μ_2, and K. *Hint:* You may use the following approximation: $$e^x \approx 1 + x + \frac{x^2}{2}, \quad x \ll 1$$	0.8pt

Part E. Fluid outflow through the wall (1.8 points)

An **incompressible** fluid characterized by the complete absence of internal friction (viscosity) and a constant density ρ is termed an **ideal** fluid. Hereafter, we shall focus our analysis on such an ideal fluid.

Consider a stream tube within a steady flow of an ideal fluid, bounded by small cross-sections S_1 and S_2, through which the fluid flows from left to right. Assume that the fluid pressure and velocity are uniform across any cross-section within this stream tube that is perpendicular to the fluid velocity.

Let the flow velocity v_1, pressure p_1, and height h_1 of this cross-section be given at S_1. Similarly, at cross-section S_2, the flow velocity v_2, pressure p_2, and height h_2 are specified. The height h is measured from a fixed reference level.

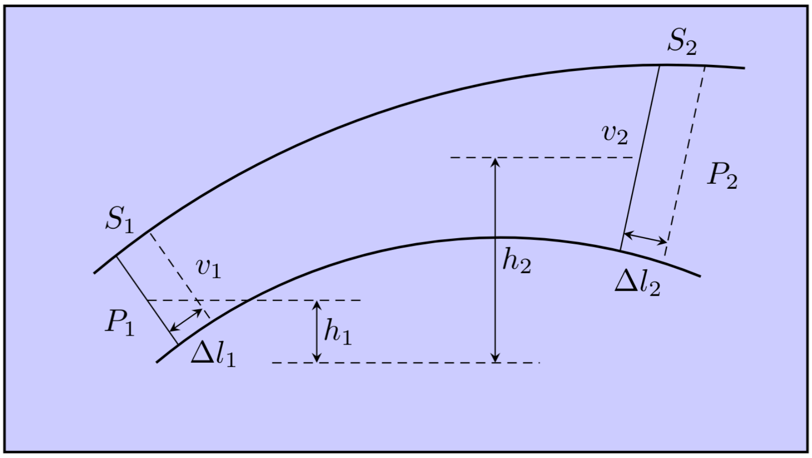

Fig. 8: Ideal fluid flow

Over a small time interval, a volume of fluid moves along the stream tube, with cross-section S_1 moving to position $S_1' \approx S_1$ over a distance Δl_1, and cross-section S_2 moving to position $S_2' \approx S_2$ over a distance Δl_2.

E.1	Derive an equation relating S_1, S_2, Δl_1, and Δl_2.	0.2pt

E.2	Determine the total energy E (the sum of kinetic and potential energies) of the selected fluid volume $S_1 \Delta l_1$. Express the answer in terms of ρ, S_1, Δl_1, v_1, g, and h_1. Take the potential energy at $h = 0$ as the zero reference level.	0.1pt

E.3	Determine the work A done by pressure forces on the fluid bounded by the chosen stream tube during time $\Delta t = \frac{\Delta l_1}{v_1}$. Express the answer in terms of p_1, p_2, S_1, and Δl_1.	0.3pt

E.4	Using the law of conservation of energy, express the pressure p_2 in terms of p_1, ρ, g, h_1, h_2, v_1 and v_2. The resulting equation is known as the **Bernoulli equation**.	0.3pt

Consider a cylindrical vessel with vertical walls, similar to the vessel in Part **B**. Its inner radius is R, and an hole with diameter $D \ll R$ is made in its wall. The vessel is filled with an ideal fluid of density ρ, and its height relative to the hole is $h \gg D$.

E.5	Determine the fluid outflow velocity $v(h)$ as a function of the height above the hole. Express the answer in terms of g and h.	0.3pt

At the initial time, let the fluid level above the hole be H.

E.6	Determine the time τ required for all the fluid initially above the hole to drain through it. Express the answer in terms of g, H, R, and D.	0.6pt

Part F. "Flow" of granules through a wall (1 point)

In this part of the problem, we examine the “flow” of granules from a cylindrical vessel, focusing on its characteristics and differences from the flow of an ideal fluid. To do this, we again consider a cylindrical vessel with vertical walls filled with granules. Assume each granule is a sphere of diameter d. A circular hole of diameter $D \gg d$ is made in the cylindrical vessel. The condition $D \gg d$ implies that the effective diameter of the opening does not decrease due to arching (accumulation of granules near it). Above the opening, there is a layer of granules of height $H \gg \lambda$, where λ is the parameter found in part **B2**. The vessel wall thickness is ω. Additionally, assume that $D \ll R$, where R is the internal radius of the vessel.

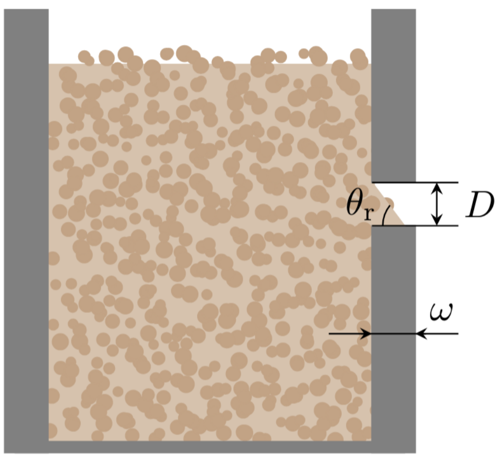

Fig. 9: Equilibrium of granules in a vessel with thick walls

F.1	Due to internal friction between the granules, the maximum angle at which the free surface of the granular material can deviate from the horizontal is θ_r. What is the maximum wall thickness ω_{max} for which the flow of granules from the vessel is possible? Express the answer in terms of D and θ_r.	0.2pt

Unlike an ideal fluid, significant energy losses occur in granular materials due to interparticle friction. Assume that the kinetic energy of the exiting particles is gained only at characteristic distances of the order of D from the hole, and the characteristic acceleration of the granules during acceleration is of the order of g.

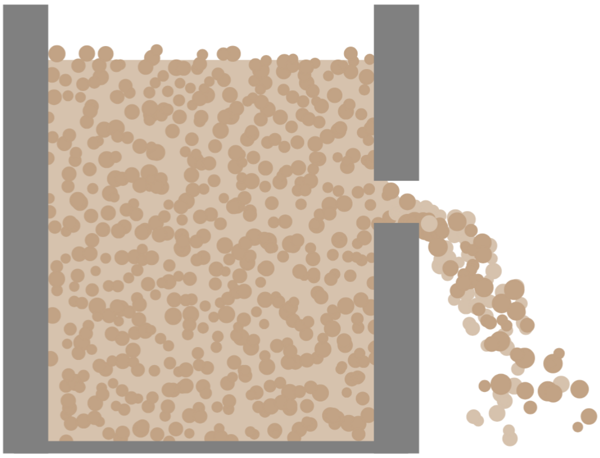

Fig. 10: Granule discharge through an orifice

Therefore, the mass flow rate of granules exiting through the hole is expressed as:

$$\frac{dm}{dt}(H, D) = AH^q D^n,$$

where A is a dimensional constant, and $H \gg D$ is the height of the layer of granules above the hole.

F.2	Determine q and n.	0.8pt