Solution

A1. Let us write the expression for specific energy at the upper point of the canal. In this point, the liquid pressure is zero, and we do not take into account the atmospheric pressure (it would give a constant addition that does not affect further calculations). Then we obtain

$$E = \frac{v^2}{2} + gd.$$

If we used a point at an random height z for the calculation, the pressure would be $P = \rho g(d - z)$, so we get the same value:

$$E = g(d - z) + \frac{v^2}{2} + gz = \frac{v^2}{2} + gd.$$

The flow velocity can be expressed in terms of the volume flow rate Q and the cross-sectional area of the flow $S = bd$,

$$v = \frac{Q}{bd}.$$

We finally obtain

$$\boxed{E = \frac{Q^2}{2b^2d^2} + gd}$$

A2. To find the minimum, we differentiate the expression for E by d:

$$\frac{\partial E}{\partial d} = -\frac{Q^2}{b^2d^3} + g = 0,$$

whence

$$\boxed{d_c = \left(\frac{Q^2}{gb^2}\right)^{1/3}}$$

A3. Let's substitute the found depth value into the formula for v:

$$v_c = \frac{Q}{bd_c} = \frac{Q}{b(Q^2/gb^2)^{1/3}} = \left(\frac{gQ}{b}b\right)^{1/3}.$$

For the Froude number we obtain

$$Fr_c = \frac{v_c}{\sqrt{gd_c}} = \frac{(gQ/b)^{1/3}}{\sqrt{g(Q^2/gb^2)^{1/3}}} = 1.$$

$$\boxed{v_c = \left(\frac{gQ}{b}\right)^{1/3}, \quad Fr_c = 1}$$

A4. The flow of water through any cross-section of the canal is the same, so:

$$Q = bd_1v_1 = bd_2v_2, \quad v_1d_1 = v_2d_2.$$

From Bernoulli's equation for a fluid that flows on a surface

$$\frac{v_1^2}{2}2 + gd_1 = \frac{v_2^2}{2} + g(d_2 + H).$$

Here it is taken into account that the height of the liquid surface in the second part of the canal is $d_2 + H$. As was shown in task **A1**, the same values would be obtained at any depth.

$$\boxed{v_1 d_1 = v_2 d_2, \quad \tfrac{v_1^2}{2} + g d_1 = \tfrac{v_2^2}{2} + g(d_2 + H)}$$

A5. Since the changes in depth and velocity are small, we can rewrite the flow conservation condition as

$$d_2 = d_1 \frac{v_1}{v_2} = d_1 \frac{1}{1 + \Delta v / v_1} \approx d_1 \left(1 - \frac{\Delta v}{v_1}\right).$$

Let's substitute this result into Bernoulli's equation:

$$\frac{v_1^2}{2} + g d_1 = \frac{1}{2}(v_1 + \Delta v)^2 + g d_1 \left(1 - \frac{\Delta v}{v_1}\right) + gH.$$

Expanding to first order by Δv and reducing the same summands, we obtain

$$v_1 \Delta v - \frac{g d_1}{v_1} \Delta v + gH = 0,$$

whence

$$\Delta v = \frac{g H v_1}{g d_1 - v_1^2}.$$

Then the change in the depth of the flow

$$\Delta d = -d_1 \frac{\Delta v}{v_1} = -\frac{g H d_1}{g d_1 - v_1^2}.$$

$$\boxed{\frac{\Delta v}{v_1} = \frac{gH}{g d_1 - v_1^2}, \quad \frac{\Delta d}{d_1} = -\frac{gH}{g d_1 - v_1^2}}$$

A6. It follows from the answer in the previous task that when $v_1^2 < g d_1$, i.e., when $Fr_1 < 1$, the flow velocity increases, and when $Fr_1 > 1$ – decreases. In the case of critical flow $Fr_1 = 1$ in the first order the velocity change goes to infinity.

$$\boxed{\text{The velocity increases when } Fr < 1, \text{ decreases when } Fr > 1}$$

B1. The pressure at height z from the bottom is $P = \rho g (d_1 - z)$ (in the left part of the system before the jump). The force that acts on the selected area on the left side can be found using the integral

$$F_1 = \int_0^{d_1} P b \mathrm{d}z = \int_0^{d_1} \rho g (d_1 - z) b \mathrm{d}z = \frac{1}{2} \rho g b d_1^2.$$

Similarly, the force acting on the right side is

$$F_2 = \frac{1}{2} \rho g b d_2^2.$$

The total force is equal to the difference of these two forces because they act in opposite directions. Consider the force positive if it acts to the right

$$\boxed{F = \frac{1}{2} \rho g b (d_1^2 - d_2^2).}$$

B2. The flow conservation condition looks exactly the same as in the previous part:

$$v_1 d_1 = v_2 d_2.$$

In time $\mathrm{d}t$, a mass of fluid $\mathrm{d}m = \rho Q \mathrm{d}t = \rho b d_1 v_1 \mathrm{d}t = \rho b d_2 v_2 \mathrm{d}t$ passes through any cross section of the flow. Then the momentum of the fluid entering the area

$$\mathrm{d}p_1 = \mathrm{d}m v_1 = \rho b d_1 v_1^2 \mathrm{d}t,$$

and the momentum of the outgoing fluid

$$\mathrm{d}p_2 = \mathrm{d}m v_2 = \rho b d_2 v_2^2 \mathrm{d}t.$$

The change of momentum is due to the force acting on the system

$$\mathrm{d}p_2 - \mathrm{d}p_1 = F \mathrm{d}t,$$

therefore

$$\rho b (d_2 v_2^2 - d_1 v_1^2) \mathrm{d}t = \frac{1}{2} \rho g b (d_1^2 - d_2^2) \mathrm{d}t.$$

Reducing the common multipliers and regrouping the summands, we obtain

$$v_1^2 d_1 + \frac{g}{2} d_1^2 = v_2^2 d_2 + \frac{g}{2} d_2^2.$$

$$\boxed{v_1 d_1 = v_2 d_2, \quad v_1^2 d_1 + \frac{g}{2} d_1^2 = v_2^2 d_2 + \frac{g}{2} d_2^2}$$

B3. From the flow conservation condition, let us eliminate the velocity of the fluid after the jump

$$v_2 = v_1 \frac{d_1}{d_2},$$

then the second equation takes the form

$$\frac{g}{2} d_2^2 + v_1^2 \frac{d_1^2}{d_2} = \frac{g}{2} d_1^2 + v_1^2 d_1,$$

After multiplication by d_2, this equation reduces to a cubic equation with respect to d_2. However, one solution to this equation $d_2 = d_1$ is known, corresponding to the fact that there is no jump and the flow of the fluid has not changed. This allows us to isolate the common multiplier $d_2 - d_1$ in the equation and reduce the problem to a quadratic equation:

$$\begin{aligned}
\frac{g}{2} d_2 (d_2^2 - d_1^2) + v_1^2 (d_1^2 - d_1 d_2) &= 0, \\
\frac{g}{2} d_2 (d_2 + d_1)(d_2 - d_1) - v_1^2 d_1 (d_2 - d_1) &= 0, \\
(d_2 - d_1) \left(\frac{g}{2} (d_2^2 + d_1 d_2) - v_1^2 d_1 \right) &= 0.
\end{aligned}$$

We are not interested in solving $d_1 = d_2$, so the multiplier $d_2 - d_1$ can be reduced, resulting in a quadratic equation

$$d_2^2 + d_1 d_2 - \frac{2 d_1}{g} v_1^2 = 0,$$

its solutions are

$$d_2 = -\frac{d_1}{2} \pm \frac{1}{2} \sqrt{d_1^2 + \frac{8 d_1 v_1^2}{g}}.$$

Since $d_2 > 0$, only the solution with the $+$ sign is suitable. We finally find

$$\frac{d_2}{d_1} = \frac{1}{2}\left(\sqrt{1+\frac{8v_1^2}{gd_1}} - 1\right) = \frac{1}{2}\left(\sqrt{1+8Fr^2} - 1\right).$$

$$\boxed{\frac{d_2}{d_1} = \frac{1}{2}\left(\sqrt{1+8Fr^2} - 1\right)}$$

B4. The difference of energies expressed in terms of velocities and depths is of the form

$$\Delta E = \frac{v_2^2 - v_1^2}{2} + g(d_2 - d_1),$$

From the quadratic equation for d_2 we find

$$v_1^2 = \frac{g}{2d_1}(d_2^2 + d_1 d_2) = \frac{gd_2}{2d_1}(d_1 + d_2).$$

Then the velocity after the jump

$$v_2^2 = v_2^2\frac{d_2^2}{d_1^2} = \frac{gd_1}{2d_2}(d_1 + d_2).$$

Substituting these results into the energy change, we obtain

$$\Delta E = \frac{g}{4}\left(\frac{d_1}{d_2} - \frac{d_2}{d_1}\right)(d_1 + d_2) + g(d_2 - d_1) =$$

$$= \frac{g(d_1 - d_2)(d_1 + d_2)^2}{4d_1 d_2} + g(d_2 - d_1) = g(d_2 - d_1)\left(1 - \frac{(d_1 + d_2)^2}{4d_1 d_2}\right)$$

We find after simplifying

$$\boxed{\Delta E = \frac{g(d_1 - d_2)^3}{4d_1 d_2}}$$

B5. In the process of motion, part of the energy can be transferred to heat due to the effects of turbulence and viscosity. At the same time, the energy cannot increase without external influence. Therefore, the condition $\Delta E < 0$ must be fulfilled, and hence $d_1 < d_2$, i.e. the depth of the flow must increase, so there is indeed a jump. Hence we get the inequality

$$\frac{1}{2}\left(\sqrt{1+8Fr^2} - 1\right) > 1,$$

which means

$$\sqrt{1+8Fr^2} > 3,$$

so $Fr > 1$.

$$\boxed{\text{The jump is possible when } Fr > 1}$$

C1. Flow in an falling jet is related to velocity by the relation

$$Q = \frac{\pi D^2}{4} v,$$

whence the velocity of water in the jet

$$v = \frac{4Q}{\pi D^2}.$$

As given, this same velocity is equal to the velocity of the stream before the jump.

$$\boxed{v = \frac{4Q}{\pi D^2} = 0.38 \text{ m/s}}$$

C2. Let us select a cylinder of radius r centered on the symmetry axis of the system. The flux through the side of this cylinder is equal to

$$Q = 2\pi r d v,$$

whence the depth of the water layer

$$d = \frac{Q}{2\pi r v} = \frac{D^2}{8r}.$$

$$\boxed{d = \frac{D^2}{8r} = 0.42 \text{ mm}}$$

C3. Using the equations for velocity and depth, we find

$$Fr = \frac{v}{\sqrt{gd}} = \frac{8\sqrt{2}}{\pi}\frac{Q}{D^3}\sqrt{\frac{r_c}{g}} = 6.0,$$

then the depth ratio

$$\frac{d_2}{d_1} = \frac{1}{2}\left(\sqrt{1 + \frac{1024 Q^2 r_c}{\pi^2 g D^6}} - 1\right) = 8.0$$

$$\boxed{Fr = 6.0, \quad \frac{d_2}{d_1} = 8.0}$$