Solution

A1. By plugging in $I = U/R(x)$ into the second Williams–Strukov equation, we obtain:

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{U}{\beta R_{\text{off}}}\frac{1}{1+(r-1)x} - \alpha x}$$

A2. The stationary value of x can be found by solving the equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0 \implies$

$$\frac{U}{\beta R_{\text{off}}}\frac{1}{1+(r-1)|x|} = \alpha x.$$

From this one can see that in the limit of large U the stationary value of x will also be large, so we can neglect 1 in the denominator. From this we get:

$$x^2 = \frac{U}{\alpha\beta R_{\text{off}}(r-1)} = \frac{U\xi}{\alpha(r-1)^2} \implies$$

$$\boxed{x_0 = \frac{9}{10(r-1)}\sqrt{\frac{U\xi}{\alpha}}}$$

A3. In the limits of large U the equation from **A1** can be rewritten in the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{U}{\beta R_{\text{off}}(r-1)}\frac{1}{x} - \alpha x \overset{y=\frac{\alpha\beta(r-1)R_{\text{off}}}{U}x^2}{\implies} \frac{\mathrm{d}y}{1-y} = 2\alpha\,\mathrm{d}t \implies$$

$$2\alpha\tau = \int\limits_0^{0.81}\frac{\mathrm{d}y}{1-y}$$

$$\boxed{\tau = \frac{\ln(100/19)}{2\alpha} \approx 0.830\alpha^{-1}}$$

A4. Minimal energy is equal to Joule heat dissipation during the off-to-on switch:

$$Q = \int\limits_0^{\tau}\frac{U^2}{R(x(t))}\,\mathrm{d}t = U^2\int\limits_0^{x_0}\frac{\mathrm{d}x}{(r-1)R_{\text{off}}x\left[\frac{U}{\beta(r-1)R_{\text{off}}x} - \alpha x\right]} \overset{z=\sqrt{\frac{\alpha\beta(r-1)R_{\text{off}}}{U}}x}{=} \frac{U^{3/2}}{R_{\text{off}}\sqrt{\alpha\xi}}\int\limits_0^{0.9}\frac{\mathrm{d}z}{1-z^2} \implies$$

$$\boxed{Q = \frac{U^{3/2}\,\mathrm{arth}\,0.9}{R_{\text{off}}\sqrt{\alpha\xi}} \approx 1.47\frac{U^{3/2}}{R_{\text{off}}\sqrt{\alpha\xi}}}$$

B1. As the term with current in the second Williams–Strukov equation is already the next order by the aforementioned quantity, we can plug in the zeroth-order approximation for current which is simply $\dfrac{U\cos\omega t}{R_{\text{off}}}$, so that:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{U\cos\omega t}{\beta R_{\text{off}}} - \alpha x.$$

Substituting $x(t) = A\cos\omega t + B\sin\omega t$, we obtain:

$$\boxed{x(t) = \frac{U}{\beta R_{\text{off}}(\omega^2+\alpha^2)}\Big[\alpha\cos\omega t + \omega\sin\omega t\Big]}$$

The difference $\delta I(t)$ between actual current and zeroth-order value $U(t)/R_{\text{off}}$ can be calculated as:

$$\delta I(t) = \frac{U\cos\omega t}{R_{\text{off}}}\left[1 - \frac{1}{1+(r-1)x}\right] = \frac{(r-1)U^2}{\beta R_{\text{off}}^2(\omega^2+\alpha^2)}\big[\alpha\cos\omega t + \omega\sin\omega t\big]\cos\omega t$$

Then:

$$\Delta I(\varphi) = |\delta I(\varphi/\omega) - \delta I(-\varphi/\omega)| \implies$$

$$\boxed{\Delta I(\varphi) = \frac{(r-1)\omega U^2}{\beta R_{\text{off}}^2(\omega^2+\alpha^2)}|\sin 2\varphi|}$$

B2. Because $\max\limits_{\varphi} |\sin 2\varphi| = 1$, and in the zeroth order $I_0 = U/R_{\text{off}} \implies$

$$\boxed{s = \frac{\omega U\xi}{2(\omega^2+\alpha^2)}}$$

B3. The dependency of $\dfrac{\omega U}{2s}$ vs ω^2 is linear with a slope of $1/\xi$ and intersept α^2/ξ. We can find the answers using the least squares method:

$f,\ \text{Hz}$	$s,\ 10^{-3}$	$\omega,\ \text{rad/s}$	$\omega^2,\ 10^6\ \text{rad}^2/\text{s}^2$	$\omega U/2s,\ 10^3\ \text{rad}\cdot\text{V/s}$
100	47.8	628	0.395	6.57
250	54.1	1571	2.467	14.52
500	36.7	3142	9.870	42.80

$$\frac{1}{\xi} = 3.82\cdot 10^{-3}\ \text{s}\cdot\text{V}^{-1}, \qquad \frac{\alpha^2}{\xi} = 5.07\cdot 10^{3}\ \text{s}^{-1}\cdot\text{V}^{-1} \implies$$

$$\boxed{\alpha = 1.15\cdot 10^3\ \text{s}^{-1}, \quad \xi = 2.62\cdot 10^2\ \text{V}^{-1}\cdot\text{s}^{-1}}$$

C1. Using the result of **A3**, we obtain:

$$\boxed{\tau = 7.2\cdot 10^{-4}\ \text{s}}$$

C2. Using the result of **A4**, we obtain:

$$\boxed{Q = 1.0\cdot 10^{-4}\ \text{J}}$$

D1. Equilibrium states can be found by solving the equation:

$$\frac{\partial\mathcal{W}}{\partial P} = 0, \qquad \frac{\partial^2\mathcal{W}}{\partial P^2} > 0.$$

As we pass the critical field, one of equilibrium states disappears, which means that the derivative $\partial\mathcal{W}/\partial P$ no longer crosses zero at this point. This means that at the critical field this derivative only touches zero, so at this point:

$$\frac{\partial^2\mathcal{W}}{\partial P^2} = 0.$$

And we conclude that one of equilibrium states disappears at the point (don't forget that we take $P < 0$):

$$\begin{cases} \dfrac{\partial\mathcal{W}}{\partial P} = -2aP + 4bP^3 - E = 0 \\ \dfrac{\partial^2\mathcal{W}}{\partial P^2} = -2a + 12bP^2 = 0 \end{cases} \implies$$

$$\boxed{P_{\text{cr}} = -\sqrt{\frac{a}{6b}}, \qquad E_{\text{cr}} = \frac{2a}{3}\sqrt{\frac{2a}{3b}}}$$

D2. We basically need to find a root of a cubic equation

$$-2aP + 4bP^3 - \frac{2a}{3}\sqrt{\frac{2a}{3b}} = 0,$$

when we know its root $P_{\text{cr}} = -\sqrt{a/6b}$ of multiplicity 2. Simlply dividing by $(P - P_{\text{cr}})^2$, we get:

$$\boxed{P_{\text{after}} = \sqrt{\frac{2a}{3b}}}$$

D3. Plugging in expressions obtained in the previous tasks:

$$\begin{aligned} \mathcal{Q} &= \mathcal{W}(P_{\text{cr}}) - \mathcal{W}(P_{\text{after}}) = \\ &= -\frac{2a}{3}\sqrt{\frac{2a}{3b}}\left(\sqrt{\frac{2a}{3b}} + \sqrt{\frac{a}{6b}}\right) - a\left(\frac{2a}{3b} + \frac{a}{6b}\right) + b\left(\left[\frac{2a}{3b}\right]^2 + \left[\frac{a}{6b}\right]^2\right) \implies \end{aligned}$$

$$\boxed{\mathcal{Q} = \frac{3a^2}{4b}}$$

D4. To write one bit of data we generally need energy of the order:

$$Q_{1\text{ bit}} = \mathcal{Q}l^3.$$

Then, to write the whole 1 TiB, we would need:

$$\boxed{Q_{1\text{ TiB}} = \frac{3 \cdot 2^{41} l^3 a^2}{b} = 6.0\text{ J}}$$