The Greenhouse Effect (10 points)

In 2021, Syukuro Manabe and Klaus Hasselmann were awarded half of the Nobel Prize in Physics for their work in modeling Earth's climate and accurately predicting the global warming caused by human industrial activities. In this problem, we will examine a simple model of global warming due to the greenhouse effect. The greenhouse gases alter the optical properties of the Earth's atmosphere in transmitting or absorbing Earth's infrared radiation, resulting in a rise in the average temperature of the planet.

All objects, at different temperatures, emit thermal radiation. The quantity $u(\lambda,T)d\lambda$ indicates the thermal radiative power per unit area of an object at temperature T between the wavelengths λ and $\lambda + d\lambda$. According to Planck's theory of blackbody radiation, we have

$$u(\lambda,T) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}, \quad (1)$$

in which $hc = 1.24 \times 10^3\ \mathrm{eV \cdot nm}$ and $k_B = 8.62 \times 10^{-5}\ \mathrm{eV \cdot K^{-1}}$. The wavelength corresponding to the maximum of $u(\lambda,T)d\lambda$ comes from the relation $\lambda_{max}T = b$ (Wien's displacement law).

Indeed, using **Eq.(1)**, it can be shown that $b = \frac{hc}{x_m k_B}$, where the dimensionless quantity x_m is the non-trivial root of an equation of the form $f(x) = 0$; you are asked to find the function $f(x)$ in one of the following tasks.

Total radiative power per unit area of a blackbody in all wavelengths is given by the Stefan-Boltzmann law as $U(T) = \sigma T^4$ where $\sigma = 5.67 \times 10^{-8}\ \mathrm{W \cdot m^{-2} \cdot K^{-4}}$. Moreover, according to Kirchhoff's law of radiation, at thermal equilibrium a body absorbing a certain fraction of the incident radiation at a specific wavelength, will radiate the same fraction of the blackbody radiation at that same wavelength.

Throughout this problem, assume that the Sun is a blackbody at its average surface temperature of $T_S = 5.77 \times 10^3\ \mathrm{K}$. The Sun's radius is $R_S = 6.96 \times 10^8\ \mathrm{m}$ and the average distance between the Earth and the Sun is $d = 1.50 \times 10^{11}\ \mathrm{m}$. We denote by $\tilde{u}_S(\lambda)$, the spectral solar power radiated into a unit area of the Earth normal to the direction of radiation. The integral of this quantity over all wavelengths, i.e. $S_0 = \int \tilde{u}_S(\lambda)d\lambda$, is called the solar constant.

In this problem, assume that the Earth is in thermal equilibrium and has the same temperature at all points on the surface. In all parts of the problem, express the desired quantity in parametric form in terms of the data given in the problem and then find its numerical value accurate to three significant figures. The required units are indicated on the answer sheet.

Part A. The Earth as a Blackbody (3.0 points)

In this part, consider the Earth's surface as a blackbody and neglect the Earth's atmosphere.

A.1	Find the solar constant S_0.	0.6 pt

A.2	Find the Earth's temperature T_E.	0.6 pt

A.3	Find the function $f(x)$.	0.4 pt

A.4	Calculate the numerical value of x_m, and from this value x_m, find the value of b	0.4 pt

A.5	Find λ_{max} for the sun and the Earth.	0.2 pt

In **Fig.1**, the functions $\gamma\tilde{u}_S(\lambda)$ and $u(\lambda, T_E)$ are plotted versus λ, where γ is a dimensionless coefficient to rescale $\tilde{u}_S(\lambda)$ such that the two peaks coincide.

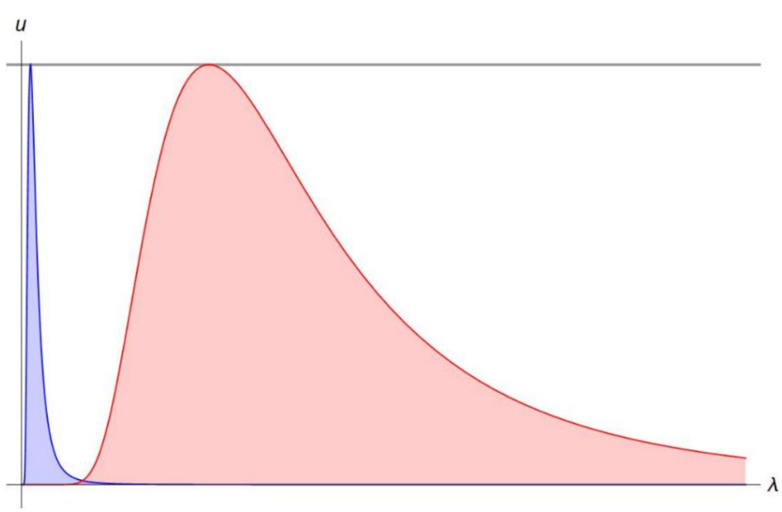


Fig. 1. The plot of $u(\lambda, T_{\mathrm{E}})$ (red) and $\gamma\tilde{u}_{\mathrm{S}}(\lambda)$ (blue) versus λ

A.6	Determine γ.	0.8 pt

Part B. The Greenhouse Effect (7.0 points)

In this part, we introduce a simple model in which the Earth's atmosphere is modeled as a thin layer at a small distance above the Earth's surface so that the difference between the area of the atmosphere's layer and the area of the Earth's surface can be neglected (see **Fig. 2**).

In what follows, assume that the major part of the thermal radiation from the Earth and the Sun are emitted at wavelengths near $\lambda_{\max}$ the for each one. Also assume that the "atmosphere layer" reflects a fraction $r_A = 0.255$ of the **visible-ultraviolet** radiation incident from above or below, and completely transmits the rest. Assume that the atmosphere does not reflect any part of the **infrared** radiation, however, it absorbs a fraction ε of the **infrared** radiation and transmits the rest. This behavior, known as the greenhouse effect, changes the average temperature of the Earth. The Earth's surface, on the other hand, reflects a fraction r_E of the **visible-ultraviolet** radiation and absorbs the rest of this radiation and all the **infrared** radiation.

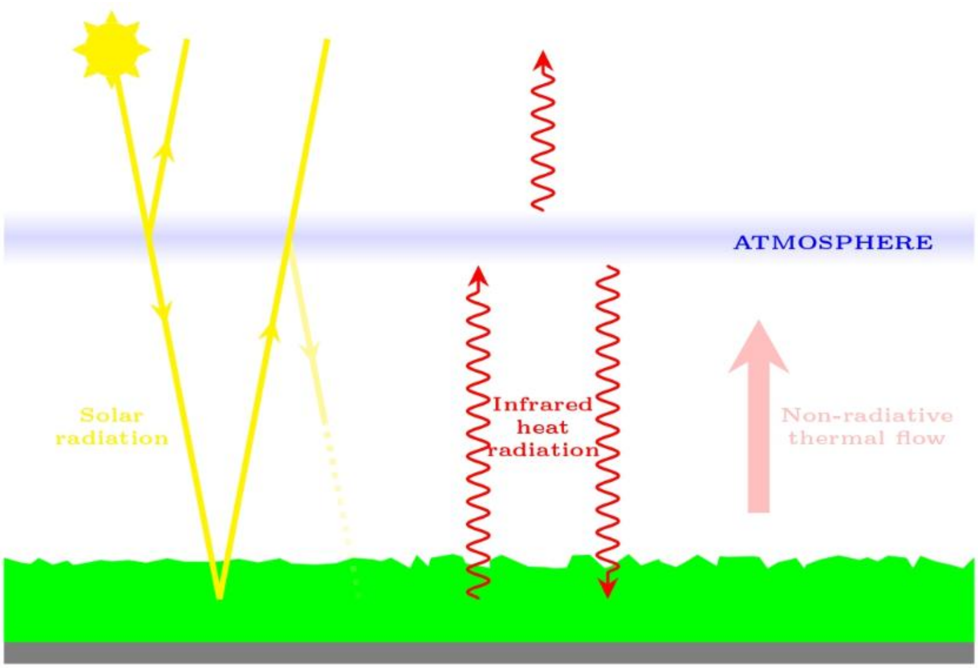


Fig. 2. Thermal flows between the Earth and the atmosphere

B.1	Assuming that $\varepsilon = 1$ and $r_E = 0$, calculate the Earth's temperature T_E and the atmosphere's temperature T_A .	1.0 pt

Now assume that $r_E \neq 0$. In this case, the combined system of "Earth + atmosphere" reflects a different fraction of the solar radiation, called "albedo" and denoted by α.

B.2	Determine the albedo, α, in terms of r_E and r_A . Then calculate its numerical value assuming $r_E = 0.102$ (and $r_A = 0.255$).	1.6 pt

B.3	a) Express the Earth's temperature in terms of σ, α, S_0, and ε. b) Using the given data and the calculated albedo, find the numerical value of ε which leads to the current average temperature of $T_E = 288\ \mathrm{K}$ for the Earth.	1.0 pt

B.4	Find $\frac{dT_E}{d\varepsilon}$ and determine by how much the Earth's temperature increases if ε increases by one percent.	1.6 pt

Assume $T_A = 245\ \mathrm{K}$ and $T_E = 288\ \mathrm{K}$. These values come from real data and may differ from the results which you have obtained in the previous tasks. Now suppose that a non-radiative (e.g. convective) thermal flow $J_{NR} = k(T_E - T_A)$ is maintained from the Earth to the atmosphere, where k is a constant. The quantity, J_{NR} , is the transmitted power per unit area.

B.5	Calculate ε and k in terms of T_E, T_A, σ, α and S_0.	1.6 pt

B.6	a) Differentiating the equations obtained in **B.5** with respect to ε, find the two algebraic equations satisfied by $\frac{dT_A}{d\varepsilon}$ and $\frac{dT_E}{d\varepsilon}$ b) Use these equations to find the numerical value of change in the Earth's temperature as a result of one percent increase in the value of ε.	1.0 pt

Trapping Ions and Cooling Atoms (10 points)

In recent decades, trapping and cooling atoms and ions has been a fascinating topic for physicists, with several Nobel prizes awarded for work in this area.

In the first part of this question, we will explore a technique for trapping ions, known as the "Paul trap". Wolfgang Paul and Hans Dehmelt received one half of the 1989 Nobel Prize in Physics for this work.

Next, we investigate the Doppler cooling technique, one of the works cited in the press release for the 1997 Nobel Prize in Physics awarded to Steven Chu, Claude Cohen-Tannoudji, and William Daniel Phillips "for developments of methods to cool and trap atoms with laser light".

Physical Constants:

mass of hydrogen atom	$m_H = 1.674 \times 10^{-27}$ kg
charge of an electron	$e = 1.602 \times 10^{-19}$ C
permittivity of free space	$\varepsilon_0 = 8.854 \times 10^{-12}\ \mathrm{F \cdot m^{-1}}$
Boltzmann constant	$k_B = 1.381 \times 10^{-23}\ \mathrm{J \cdot K^{-1}}$
reduced Planck constant	$\hbar = 1.055 \times 10^{-34}\ \mathrm{J \cdot s}$

Part A. The Paul Trap (5.6 points)

It is known that with electrostatic fields, it is not possible to create a stable equilibrium for a charged particle. Therefore, creating a stable equilibrium point for ions requires more sophisticated techniques. The Paul trap is one of these techniques.

Consider a ring of charge with a radius R and a uniform positive linear charge density λ . A positive point charge Q with mass m is placed at the center of the ring.

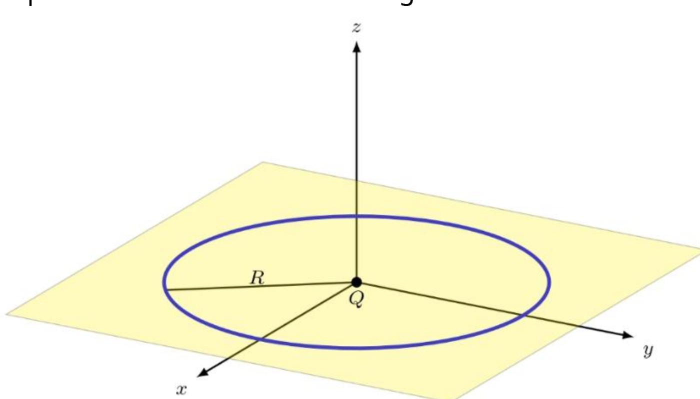


Fig. 1. A positively charged ring with a uniform linear charge density λ and radius R : The origin of the coordinate system is at the center of the ring.

A.1	a) In cartesian coordinates (x, y, z), obtain the electric field due to the charged ring in the vicinity of the ring's center to the first order in x/R, y/R, and z/R. b) Find the angular frequency of small oscillations of the charged particle around the center of the ring in the directions for which a stable equilibrium exists.	1.5 pt

In order to trap the charge Q fully, we would like to apply alternating fields to produce a dynamic equilibrium. Assume that the charge density is $\lambda(t) = \lambda_0 + u \cos \Omega t$ in which λ_0, u, and Ω are adjustable. We shall ignore radiative effects. Then the equation of motion for small displacements from the center of the ring, along the direction perpendicular to the plane of the ring will turn out to be:

$$\ddot{z} = (+k^2 + a\Omega^2 \cos \Omega t)z \quad (1)$$

A.2	Write a and k in terms of the known parameters	0.4 pt

We would like to obtain an approximate solution to **Eq.(1)** by making the following simplifying assumptions: $a \ll 1, \Omega \gg k$, and $a\Omega^2 \gg k^2$. With these assumptions, it can be shown that the solution of this equation can be split into two parts: $z(t) = p(t) + q(t)$, where $p(t)$ is a slowly varying component and $q(t)$ is a **small-amplitude** rapidly-varying component with a **mean value of zero**. In other words, $p(t)$ may be assumed constant over a few oscillations of $q(t)$ (see **Fig. 2.**)

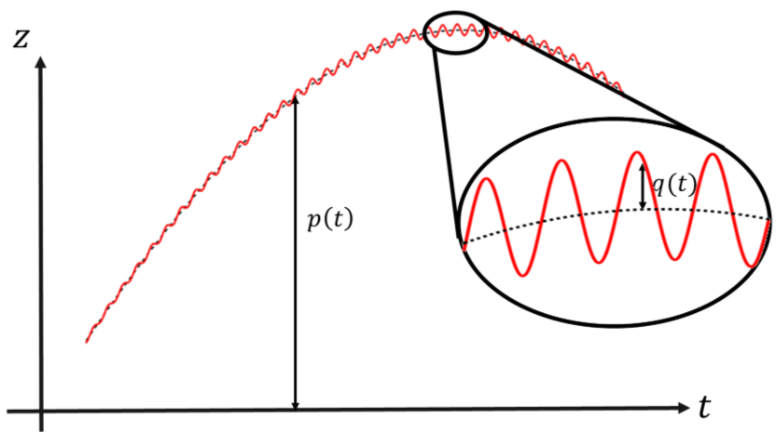


Fig. 2. A typical solution for the equation of motion of the charged particle: $p(t)$ gives the overall motion, and $q(t)$ represents small oscillations around this trajectory. The ellipse on the right is a magnification of a part of this trajectory.

A.3	a) Using the approximations stated above, find the equation of motion for $q(t)$ in terms of a, Ω, and $p(t)$. b) Find the solution of this equation by considering appropriate initial conditions corresponding to the required properties of this function	1.8 pt

A.4	a) Using the mean effect of the rapidly varying component and obtain an effective equation of motion for $p(t)$. b) Investigate the stability of the equilibrium point and find the condition for a stable equilibrium.	1.5 pt

Assume that $\lambda_0 = 8 \times 10^{-9}\ \mathrm{C \cdot m^{-1}}$ and $R = 10\ \mathrm{cm}$. We would like to use this device to trap a singly ionized atom 100 times heavier than a hydrogen atom.

A.5	Calculate k. Assume $a = 0.04$ and estimate the smallest frequency required to stabilize the motion of this ion. Use the data given at the start of the question.	0.4 pt

Part B. Doppler Cooling (4.4 points)

It may be necessary to cool a trapped atom or ion. Assume that a trapped atom of mass m , has two energy levels with an energy difference of $E_0 = \hbar\omega_A$. Electrons in the lower level may absorb a photon and jump to the higher level, but after a period τ they will return to the lower level and emit a photon with a frequency predominantly within $[\omega_A - \Gamma, \omega_A + \Gamma]$.

B.1	Use the Heisenberg's uncertainty principle to find Γ.	0.5 pt

With a similar reasoning, when we shine a laser light on the trapped atom, if the angular frequency of the laser, ω_L, falls in the interval $[\omega_A - \Gamma, \omega_A + \Gamma]$, the atom may absorb the photon.

Assume that the frequency ω_L of the laser light is slightly lower than ω_A. For a particular device, the rate of photon absorption by an atom in the reference frame of the atom is given in **Fig. 3**. The absorbed photon is then re-emitted in a random direction.

To make things simple, we consider the problem in one dimension, i.e. we assume that the atoms can only move in the x-direction and the laser light shines on them both from the left and from the right. In the atom's reference frame, the light has a higher or lower frequency due to the motion of the atoms.

Since the velocity v of the atoms is very small, we only include terms of order $\frac{v}{c}$ and ignore all the higher-order terms. Moreover, we have $m \gg \frac{\hbar\omega_A}{c^2}$ so that the velocity of the atom nearly does not change after absorbing the photon.

Also, the change in frequency due to the Doppler effect is so small compared to $\omega_A - \omega_L$, that the function for s in the diagram of **Fig. 3** may be approximated by the following linear function:

$$s(\omega) \approx s_L + \alpha(\omega - \omega_L)$$

where s is the number of absorbed photons per unit of time, s_L is the value of s for $\omega = \omega_L$, and α is the slope of the line tangent to the curve at ω_L.

The frequency of the re-emitted photon is almost equal to the frequency of the incident photon, but it is emitted with equal probability in the positive or negative x -direction. In fact, up to the order considered here the two frequencies are identical. Note that we are considering the whole process in the atom's reference frame.

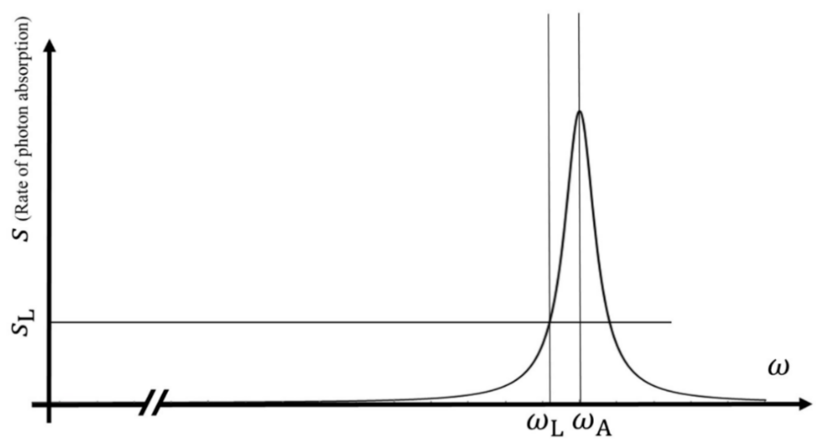


Fig. 3. The rate of photon absorption as a function of the frequency for a particular trap: the frequency corresponding to the energy difference between the two atomic levels is indicated by ω_A and the slightly smaller frequency of the laser is indicated by ω_L.

B.2	a) Assume that the trapped atom is moving with a velocity $v = v_x$, in the lab frame. In the frame of reference of the atom, calculate the collision rate of the photons, incident from each of the two directions, with the atoms (denoted by s_+ and s_-), and the rate of absorption of momentum in each direction (denoted by π_+ and π_-). b) Determine the effective force on the atom as a function of v, $k_L = \frac{\omega_L}{c}$, $\hbar$, and α, in the reference frame of the laboratory. Assume $s_L \ll \alpha\omega_L$	1.7 pt

We would like to find the lowest temperature that can be achieved using this technique. Assume that the velocity of a particular atom has been reduced to zero exactly, and at this very moment it absorbs a photon (incident from any of the two directions), and re-emits the photon randomly in any of the two directions, with almost the same frequency. Assume that this process happens once every τ units of time

B.3	Considering the momentum of the atom after such a process for the two possible outcomes, calculate the average power absorbed by the atom.	1.0 pt

B.4	Consider the force calculated in **B.2** and calculate the output power. Then, calculate the average value of v^2 at equilibrium. Using your knowledge of the kinetic theory of gases, estimate the temperature of the atoms.	0.8 pt

B.5	Estimate this temperature, for an atom 100 times heavier than a hydrogen atom. Assume that $\omega_L = 2 \times 10^{16}\ \mathrm{rad \cdot s^{-1}}$, $\tau = 5 \times 10^{-9}$ s, and $\alpha = 4$.	0.4 pt

Black Widow Pulsar (10 points)

A significant number of the observed stars are binaries. One or both of the stars may be neutron stars rotating with a high angular velocity and emitting electromagnetic waves; such stars are called pulsars.

Sometimes a companion star is an expansive mass of gas that gradually falls down onto the neutron star and causes its mass to increase (**Fig. 1(a)**). In this way, a neutron star gradually swallows up a portion of the mass of its companion star. For this reason, the neutron star has been compared to a black widow (or redback spider), a female spider which eats its mate after mating.

The heating of the gas falling down onto the black widow generates radiation which can be observed. The heaviest neutron stars often are black widows and they serve as natural laboratories for testing fundamental physics. **Fig. 1(b)** shows the picture of the companion of the neutron star PSR J2215+5135, taken by the 3.4-meter optical telescope of the Iranian National Observatory. No neutron star can be seen in this image and the observed light is due to its companion.

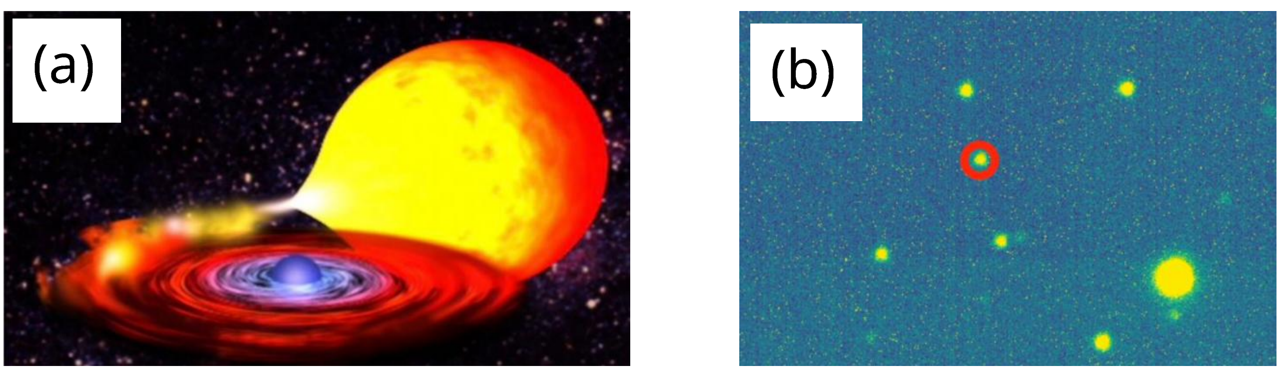


Fig. 1. **(a)** The falling of the companion star onto the neutron star

(b) The companion of the neutron star PSR J2215+5135

Part A. A binary system (5.0 points)

Consider a simple model in which the black widow and its companion star, are represented by two point masses M_1 and M_2 moving on a circular orbit around their center of mass. To investigate the dynamics of this system, consider a rotating coordinate system in which the two bodies are stationary. Take the center of mass to be the origin of the coordinates system. Assume that the two point masses lie on the x-axis on both sides of the origin at a distance a from each other, and that M_1 lies on the negative x-axis. At an arbitrary point (x, y) in the plane of motion, the effective potential $\varphi(x, y)$ for a unit test mass is the sum of the gravitational potentials of the two point masses plus the centrifugal potential.

A.1	Write $\varphi(x, y)$ in terms of M_1, M_2, G, and a.	1.0 pt

A.2	Assuming $M_1 > M_2$, plot the function $\varphi(x, 0)$ qualitatively.	0.7 pt

Suppose (just for task **A.3**) $M_2 = \frac{M_1}{3}$ and assume that M_2 is surrounded by a rarefied gas of very low density. The mass of this gas is insignificant and we ignore its gravitational effects. If the size of this gas envelope becomes greater than a specific limit, the gas with overflow onto M_1. Suppose the overflow occurs through $x = x_0$ on the x-axis.

A.3	Find the numerical value of $\frac{x_0}{a}$, up to two significant figures. You may use the calculator.	0.5 pt

Take the rotational period of the stars around their center of mass to be P. Assume that mass flows from M_2 to M_1 at a very small rate of $\frac{dM_1}{dt} = \beta$. This rate is so small that in each period of rotation, the distance between the two stars can be assumed to be constant. However, after a long period of time, the distance between the two stars changes, while the motion remains circular.

A.4	Calculate the rate of change of a and P in terms of β, M_1, M_2, G, and a.	0.6 pt

The gas separated from M_2 forms a disk rotating around M_1 and heats up due to friction (**Fig. 1(a)**). As the gas loses energy, it spirals inward toward M_1 and finally falls onto it. In the steady state, the mass flows at the constant rate of β, from M_2 to the disc and from the disc onto M_1. At the same time, the heated disk emits thermal radiation as a blackbody. This disk forms very close to the neutron star so the gravitational pull of the M_2 star can be ignored for the analysis of the disk's motion. Also, ignore the heat capacity of the gas.

A.5	Determine the temperature of the disk at distance r from the center of the star M_1 in terms of β, M_1, G, and σ (Stefan-Boltzmann constant).	1.0 pt

In the binary system PSR J2215+5135, the mass of the neutron star is $M_{NS} = 2.27 M_\odot$ and the mass of its companion star is $M_S = 0.33 M_\odot$, where $M_\odot = 1.98 \times 10^{30}\ \mathrm{kg}$ is the mass of the Sun. The rotational period is $P = 4.14\ \mathrm{hr}$, the Stefan-Boltzmann constant is $\sigma = 5.67 \times 10^{-8}\ \mathrm{W \cdot m^{-2} \cdot K^{-4}}$, and the gravitational constant is $G = 6.67 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$. Assume that the mass flow rate to the neutron star is $\beta = \dot{M}_{NS} = 9 \times 10^{-10} M_\odot\ \mathrm{yr}^{-1}$.

A.6	Calculate the temperature of the disk at the radius $r = \frac{a}{10}$ in kelvins	0.5 pt

Assume that after a sudden explosion, the M_1 star ejects a part of its mass out of the binary system at a very high speed, and its mass becomes M_1' .Take the magnitude of the velocity of M_1' relative to M_2 to be v' after the explosion.

A.7	Determine the maximum value of v', in terms of M_1', M_2, G, and$a$, that allows the new binary system to stay bounded. Assuming that the explosion is isotropic, what is the minimum value of M_1' for the binary system to remain bounded?	0.7 pt

Part B. Analysis of the Stability of a Star (5.0 points)

In this part we study the stability of a single star. Consider a specific kind of matter with the equation of state $p = K\rho^\gamma$ where K and γ are constants. Let $p(r)$ and $\rho(r)$ be the pressure and density at a distance r from the center of the star, respectively. The pressure and density at the center of the star are p_c and ρ_c, respectively. In all tasks of this section, take all outward vectors to be positive.

B.1	Determine the gravitational acceleration $g(r)$ near the center of the star in terms of r and the constants G and ρ_c.	0.2 pt

B.2	Derive a (differential) equation for determining $\rho(r)$ at equilibrium, and write it in the following form: $$\frac{d}{dr}\left[h_1(\rho,r)\frac{d\rho}{dr}\right] + h_2(r)\rho = 0.$$ Find the functions h_1 and h_2.	0.6 pt

B.3	Construct a quantity r_0 of the form $r_0 = G^l p_c^m \rho_c^n$ with the dimension of length.	0.4 pt

B.4	Rewrite the (differential) equation of **B.2** in the following form: $$\frac{d}{dx}\left[A_1(u,x)\frac{du}{dx}\right] + A_2(x)u(x) = 0$$ Where $x = r/r_0$ and $u = \rho/\rho_c$. Find the functions $A_1(u,x)$ and $A_2(x)$.	0.3 pt

B.5	For $\gamma = 2$ one finds $u(x) = \frac{f(x)}{x}$. Determine $f(x)$.	0.6 pt

Assume that for a particular star $\frac{du}{dx}$, as a function of x, is given by the curve given in **Fig. 2.**

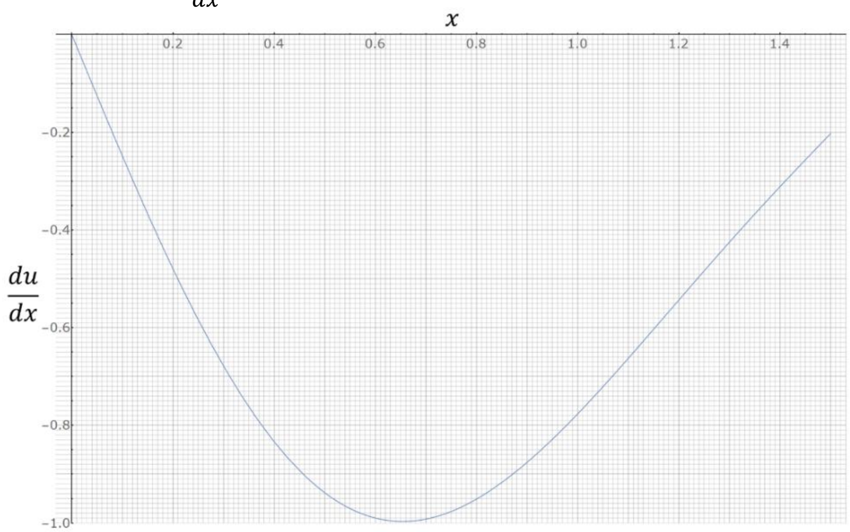


Fig. 2. The plot of $\frac{du}{dx}$

B.6	Use the behavior of the curve in **Fig. 2**, in the vicinity of the point $x = 0$, to find γ up to three significant figures. Use the given ruler if necessary.	0.8 pt

To analyze the stability of the system, we assume that the star deviates slightly from its equilibrium state: we assume that the spherical shell, which was in equilibrium at radius r, now has radius $\tilde{r}$, similarly the parameters g, p, and ρ have changed to $\tilde{g}$, $\tilde{p}$, and $\tilde{\rho}$ respectively. For convenience, we shall only consider small r's near the center of the star, for which we can assume that $\tilde{r} = r\big(1 + \varepsilon(t)\big)$, where $\varepsilon \ll 1$.

B.7	Find $\tilde{\rho}$ and $\tilde{g}$ in terms of ρ and g to the first order in ε.	0.9 pt

B.8	Using Newton's equation of motion for the spherical layer with the equilibrium radius of r, find $\frac{d^2\tilde{r}}{dt^2}$ in terms of $\tilde{g}$, $\tilde{\rho}$, K, γ, and $\frac{\partial\tilde{\rho}}{\partial\tilde{r}}$ (By $\frac{\partial\tilde{\rho}}{\partial\tilde{r}}$ we mean the derivative of $\tilde{\rho}$ with respect to $\tilde{r}$ at constant t.)	0.6 pt

B.9	Obtain $\frac{d^2\varepsilon}{dt^2}$ in terms of ε and the constants given in the problem. Find the minimum value of γ for a stable equilibrium, and find the oscillation's angular frequency of the star.	0.6 pt

Heat Conduction in a Copper Rod (10 points)

The Experimental Setup

In this experiment a 57.0 cm long copper rod with a diameter of 1.20 cm has been placed across the length of a metal box which is supported by square flanges (**Fig. 1**). The metal box serves to isolate the airflow inside the box.

Fig. 1. General view of the experimental equipment

As shown in **Fig. 2**, 7.0 cm from each side of the two ends of the rod is insulated by styrofoam. Seven holes have been drilled in the rod at equal distances of 7.0 cm; each hole has a depth of 0.6 cm and a temperature sensor (a thermistor) has been placed inside each drilled hole. These sensors have been numbered 1 through 7, from left to right. A similar sensor, numbered 8, has been placed within the box to monitor the ambient temperature of the box (θ_b).

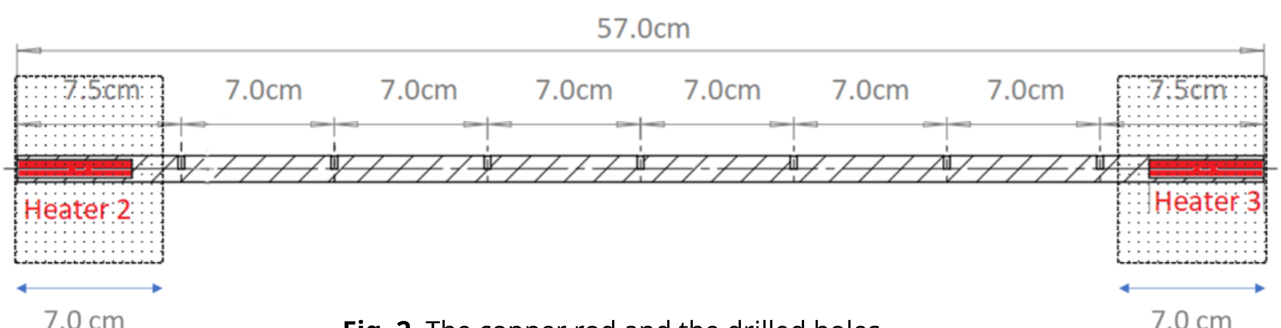


Fig. 2. The copper rod and the drilled holes

At the two insulated ends of the rod, two electric heaters with different output powers have been installed inside 5.2 cm-long holes, drilled longitudinally. The left and right heaters are designated as Heaters nos. 2 and 3, respectively. Four fans at the top of the metal box, control the air flow within the box and around the rod. These fans, as well as the copper rod, can be seen from below as shown in **Fig. 3**.

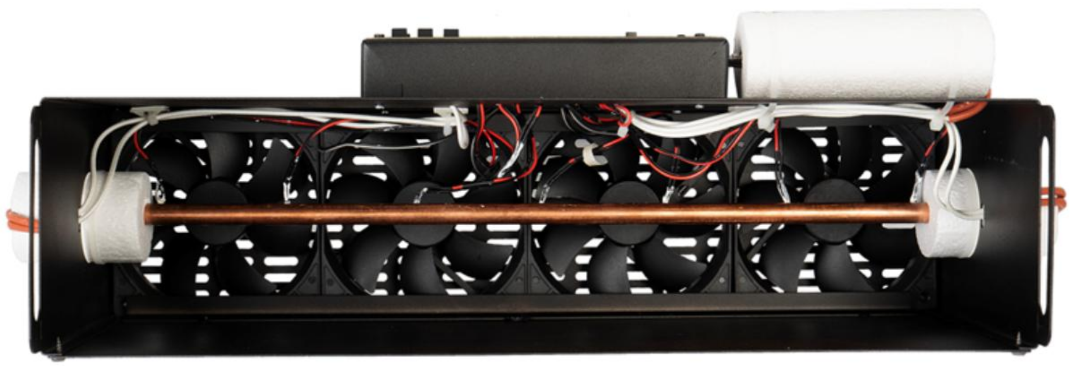


Fig. 3. View from below

A control panel, which we call AVA-T403, has been installed on the front. All the switches for turning on the various components of the experiment: the fans, the heaters, the timer, etc. are on this panel and all the measurements made by the sensors are displayed on its monitor. The power necessary to run all the instruments is provided by a 24 V power supply when connected to the power outlet. It lies next to the box. To the right of the control panel, a smaller 12.0 cm copper rod is installed; it is also covered with styrofoam. Under the styrofoam, Heater no. 1 with an output power of $1.95 \pm 0.06\ \mathrm{W}$ has been inserted inside a longitudinal hole at the right as shown in **Fig. 4**. At the other end, another longitudinal hole has been drilled in which an PT100 (Platinum Thermometer 100) thermistor bundled together with another thermistor (R_9) (similar to the other thermistors) have been inserted. These two sensors and the heater are connected to AVA, and AVA indicates the values of their resistance (in Ω)

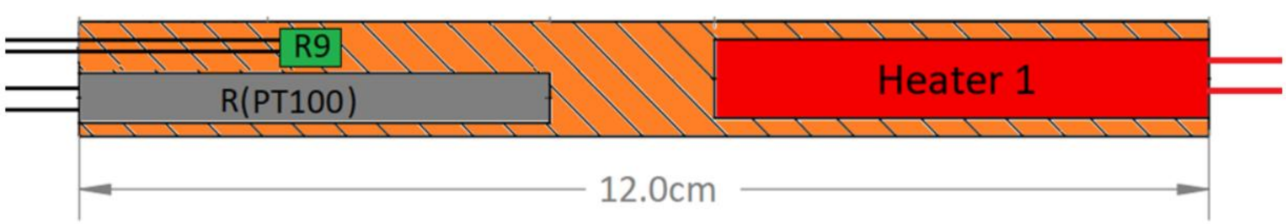


Fig. 4. The smaller copper rod

Please observe the following points:

1. Do not touch and activate the buttons on the equipment before it is asked you to do so.
2. You do not need to do anything regarding the setup. Take care not to disturb the setup and not to disconnect any wires.
3. Don’t move the setup during the experiment.
4. If the message “Turn off Heater1” appears on the monitor, immediately turn off Heater No.1
5. Turning on the heaters will increase the temperature, so it will take extra time for the system to reach its steady state, make sure that you do not turn on a heater unnecessarily.
6. Errors need to be calculated and reported whenever the $\pm$ sign is present in the answer sheet.
7. Regression, (denoted by reg in the formula below and shown as r on the calculator), is a number between 1 and -1 showing how much the data can be fitted to a line. If $|\mathrm{reg}| = 1$ it means that the data are completely on a line.

We can use the following formula to calculate the uncertainty of the slope b:

$$\Delta b = b\sqrt{\frac{1}{n-2}\left(\frac{1}{\mathrm{reg}^2} - 1\right)} ,$$

in which n is the number of data points.

The Equipment

1. Generally, Electrical resistors have different behaviors in response to a change in temperature. One of the widely used resistors is PT100 which has a linear behavior over a considerable range of temperatures, i.e.

$$R(\theta) = R_0(1 + \alpha\theta) \tag{1}$$

in which R_0 is the value of the resistance at $0\ ^\circ\mathrm{C}$, α is a constant coefficient (within the range of temperatures for this problem), and is θ the temperature in degrees Celsius. The value of α for the resistor used in this problem is $0.0039083\ ^\circ\mathrm{C}^{-1}$. For PT100, $R_0 = 100.00\ \Omega$.

2. The thermistors 1 through 9 have a nonlinear behavior in response to changes in temperature, and are usually used for measuring small changes in temperature. The resistance of these thermistors changes with temperature as follows:

$$R' = R'_0 \exp\left(\frac{E_g}{2k_B T}\right) \tag{2}$$

Where is R'_0 a constant, $k_B = 8.61733 \times 10^{-5}\ \mathrm{eV} \cdot \mathrm{K}^{-1}$ is the Boltzmann constant, T is the temperature in kelvins and E_g is the energy gap for the thermistor's semiconductor material. Remember that $T = (\theta + 273.15)\ \mathrm{K}$

3. The digital device AVA was designed by Iranian engineers specifically for this experiment. AVA measures the instantaneous resistances of Thermistors 1 through 7, PT100, and θ_b or R_9 . every two seconds. Then, based on the formula 2, reports the temperature of these sensors. However, for Sensor no. 9, only the value of the resistance is displayed. On the left of AVA's monitor, the temperatures of sensors 1 through 7 are displayed in a column. The last row, however, only shows at each instance, either the temperature of sensor 8, or the resistance of Sensor 9 (R_9). You can toggle between these two values by pressing $\theta_b - R_9$ the button shown in **Fig. 5**.

   AVA also has a timer, which works very much like any commercial timer: by pressing the Start/Stop button, the timer starts measuring the time elapsed, and pressing the Start/Stop button again stops the timer. While the timer is working, pressing the Lap button will result in all the displayable values being saved. Pressing the same button when the timer is not working resets the timer, however, the saved data will not be erased. To erase the data the Lap/Reset button has to be pressed and held for 5 seconds.

   The saved data can be seen by repeatedly pressing the Next/Prev button: pressing Prev shows older data, pressing Next shows newer data. All saved data will be erased in case the device is switched off and on. The device will not turn off automatically

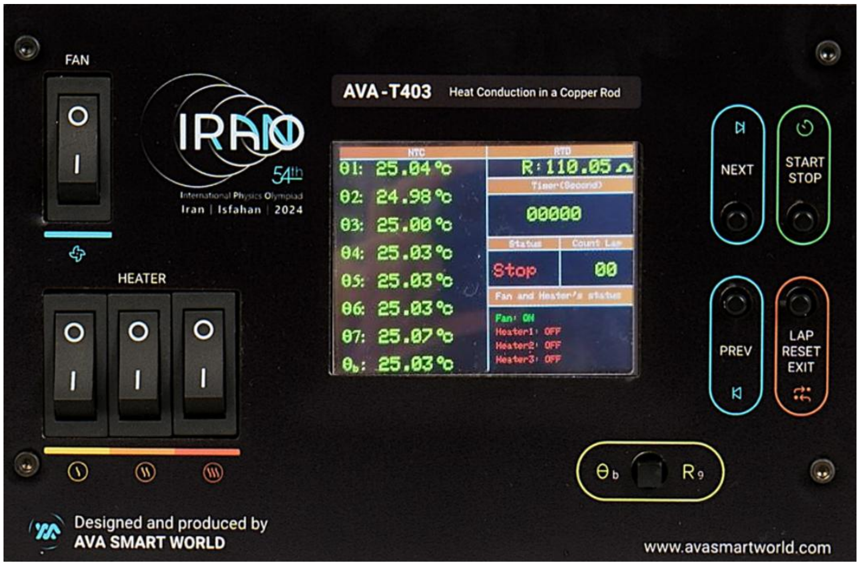


Fig. 5. Start/Stop and Lap buttons. (Further details of the key's functions come as point 3, on the page 4.)

4. As shown in **Fig. 5**, there is one switch for turning the fans on and off ( ) and there are also three switches for turning on the heater ( ). The icon for each switch is stamped below it. The first heater is a 1.95 W heater, the power of second heater is written on the device as shown in **Fig. 6,** and the power of third heater is unknown

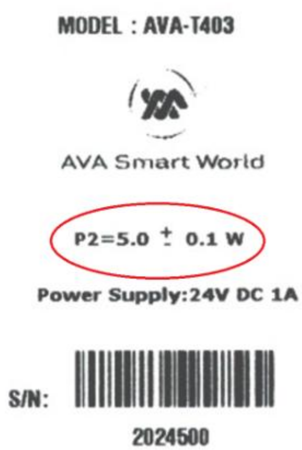


Fig. 6. The power of the second heater

Theoretical Background

In this problem, heat is transferred inside the rod through conduction, and transferred from the rod to the surrounding air through natural or forced convection. Also, due to the heat capacity of the rod, some of the heat injected into the rod is used up to raise the temperature of the rod.

a) **Heat conduction:** for a heat conductor in the shape of a rod with no heat loss from its lateral surface, the rate of heat transfer, dQ/dt , through a differential element (**Fig. 7**) at the steady state is as follows

$$\frac{dQ}{dt} = -kA\frac{d\theta}{dx} \tag{3}$$

where is the temperature difference between the two edges of the differential element, is the cross-sectional area, is the length of the differential element, and is the heat transfer coefficient (heat conductivity) which depends on the type of material the rod is made of.

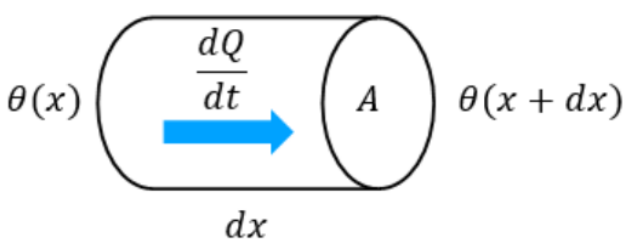


Fig. 7. The differential heat conductor element. (Heat conduction, in the absence of convection)

b) **Heat convection:** For any object exchanging heat with the air through its lateral surface, the following relationship holds:

$$\frac{dQ}{dt} = -hS\Delta\theta \tag{4}$$

in which S is the area of the lateral surface, $\Delta\theta$ is the temperature difference between the object and the surrounding air, and h is the convective heat transfer coefficient which is a function of the shape of the object and the nature of the heat flow through the sides of the object.

The Experiment:

In order to save time, we recommend that you turn on Heater 2 and the fans which are needed in Part B. Make sure that the other heaters are not turned on.

Part A. The Short Copper Rod (3.9 points)

A.0	Write numbers 0 to 9 in the table	0.0 pt

Before turning on Heater 1, the small rod is at the same temperature as its environment. Tasks **A.1** to **A.3** are related to the heating process and tasks **A.4** to **A.7** corresponds to the cooling process of the rod.

A.1	Record the initial value of resistance R_{env} (resistance of PT100) when it is at the temperature of its environment. Using **Eq.1**, find this temperature.	0.2 pt

Let us denote the total heat capacity of the rod and the heater and the sensors by C_S. To find C_S we should turn on Heater 1 and measure the change in the value of resistance for at least 150 seconds. Note that there is a time delay in the heating and cooling of the sensors.

A.2	In time intervals of approximately 10 seconds record the value of R. Do this at least 15 times.	0.5 pt

A.3	Draw a graph for **A.2**, fit a line to your data and find its slope. Using the slope find C_S.	0.8 pt

Wait until the value of R reaches $120\ \Omega$ and then turn off Heater 1. The temperature of the rod and the resistance R will start to decrease slowly after a few seconds. When PT100 is cooling down, the resistance is given by

$$R - R_{env} = Ae^{-\gamma t} \tag{5}$$

in which A and γ are constants.

A.4	At several different instances of time, measure the value of $R - R_{env}$.	0.5 pt

A.5	Make a semi-logarithmic plot of your data in **A.4**. Then find γ.	0.7 pt

The insulator around the rod causes the resistors to reach thermal equilibrium sooner, and the rod to have a more uniform temperature profile.

A.6	Measure and record the resistance R_9 of Thermistor 9 in terms of R. Measure at least seven different values of R , preferably in the $114 - 120\ \Omega$ range.	0.5 pt

A.7	Draw a graph of the resistance R_9 versus $1/T$ on the given semi-logarithmic graph and find the magnitude of the energy gap E_g in units of **eV**	0.7 pt

Part B. The Long Copper Rod (4.1 points)

Wait for the rod to reach a steady state i.e. the measured temperatures at all points remain constant, and then answer the following questions. We shall denote the temperatures of Thermistors 1-7 by θ_1 through θ_7 respectively. The location of Thermistor 1 corresponds to $x = 0$.

B.1	Record the initial value of resistance R_{env} (resistance of PT100) when it is at the temperature of its environment. Using **Eq.1**, find this temperature.	0.4 pt

In order to save time, take a look at **Part C** then continue **Part B**.

B.2	On semi-log paper, draw a diagram for the difference between the temperature of the box and the temperature at point x, $(\theta_x - \theta_b)$ along the length of the rod.	0.4 pt

It can be shown that as a function of the distance from Heater 2 along the length of the line, the temperature obeys the following relation:

$$\theta_x = \theta_b + Ae^{-\lambda x} + Be^{\lambda x} \tag{6}$$

in which θ_b the ambient temperature of the box, A and B are constants, and $\lambda = \sqrt{\frac{2h}{kr}}$,where r is the radius of the rod. As a first step, we can ignore the data corresponding to large values of x and take to be zero. In this case we can find λ and A to a first approximation, let us call them $\lambda^{(0)}$ and $A^{(0)}$:

B.3	Use the temperatures θ_1 through θ_5 and find $\lambda^{(0)}$ and $A^{(0)}$ Using the graph of **B.2**.	0.6 pt

The temperature at the end of the rod farthest from the heater does not change with x. Assume this happens around a distance $x = d$. One can use this to determine B in terms of λ, A, and d:

B.4	Express B in terms of λ, A, and d. And for $d = 44.0\ cm$, find its numerical value using the results of **B.3** Denote this quantity as $B^{(1)}$	0.4 pt

Now we can use the value obtained for B to correct the previous calculation. To do so, assume:

$$\theta'_x = \theta_x - B^{(1)} e^{\lambda^{(0)} x}$$

B.5	Find θ'_1 through θ'_7 and complete the columns added to **Table B.1**.	0.4 pt

B.6	Draw a new graph to obtain the values for λ and A. Denote them by $\lambda^{(1)}$ and $A^{(1)}$ respectively.	1.0 pt

To obtain accurate approximations, the corrections should be repeated many times, but in the end, we'll find that the final answer is close to $\lambda = \frac{\lambda^{(0)}+\lambda^{(1)}}{2}$ and $A = \frac{A^{(0)}+A^{(1)}}{2}$.

B.7	By balancing the input and output powers of the copper rod, find h and k.	0.9 pt

Part C. Measuring the Unknown Power (2.0 points)

While the fans are on, turn on Heaters 2 and 3, and wait until the temperature reaches equilibrium at all points of the copper rod. (This will take about 15 minutes.)

C.1	Measure and record the temperatures θ_1 through θ_7	0.4 pt

It can be shown that, in this case, the temperature in terms of varies as follows

$$\theta_x - \theta_b = A' \cosh(\lambda(x - x_0)) \tag{7}$$

in which A' is a constant and $cosh\ (u)$ is the hyperbolic cosine function of u defined as:

$$\cosh u = \frac{e^u + e^{-u}}{2} \tag{8}$$

C.2	Draw the graph of temperature versus distance and find x_0.	0.6 pt

C.3	Using your own method to find the effective power of Heater 3. Explain your method as clearly as possible by writing down explicitly the mathematical formulas you have used to arrive at your results.	1.0 pt

Diffraction from Phase Steps (10 points)

The Equipment Box

Fig. 1. The equipment box

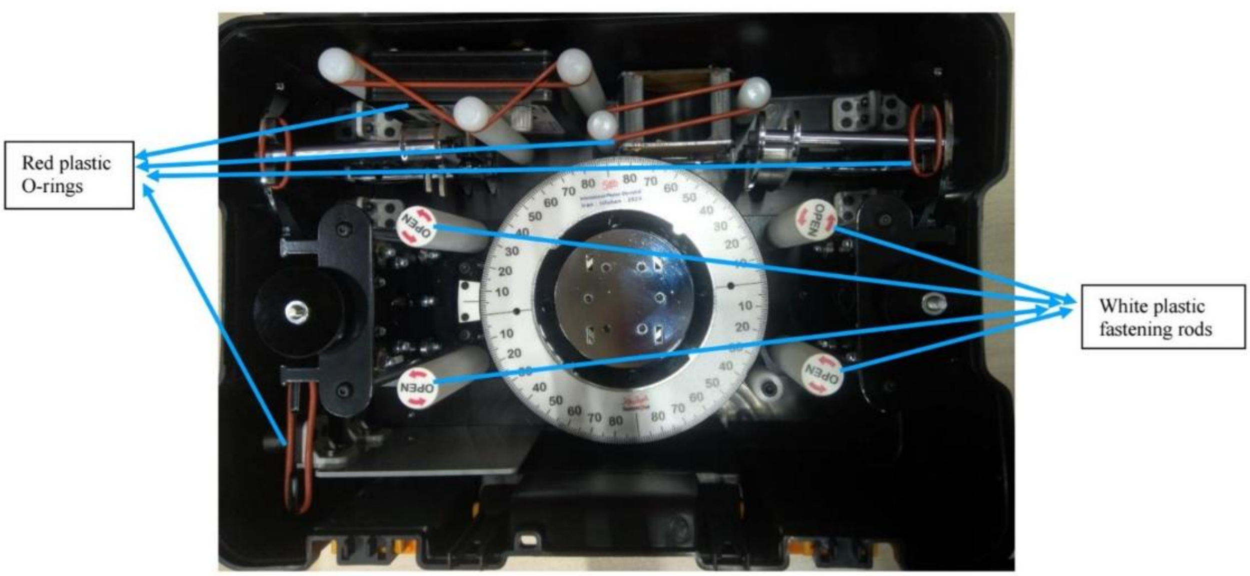


Fig. 2. Inside the box, top view

To release the setup, you need to unscrew the white plastic fastening rods in the direction indicated on the top. You can pull out the main platform from the box as shown in **Fig. 3**. After this, remove the red plastic O-rings indicated in **Fig. 2** and remove the other components inside the box one by one. To remove the instruments hold them by their metallic parts or by their outer surface.

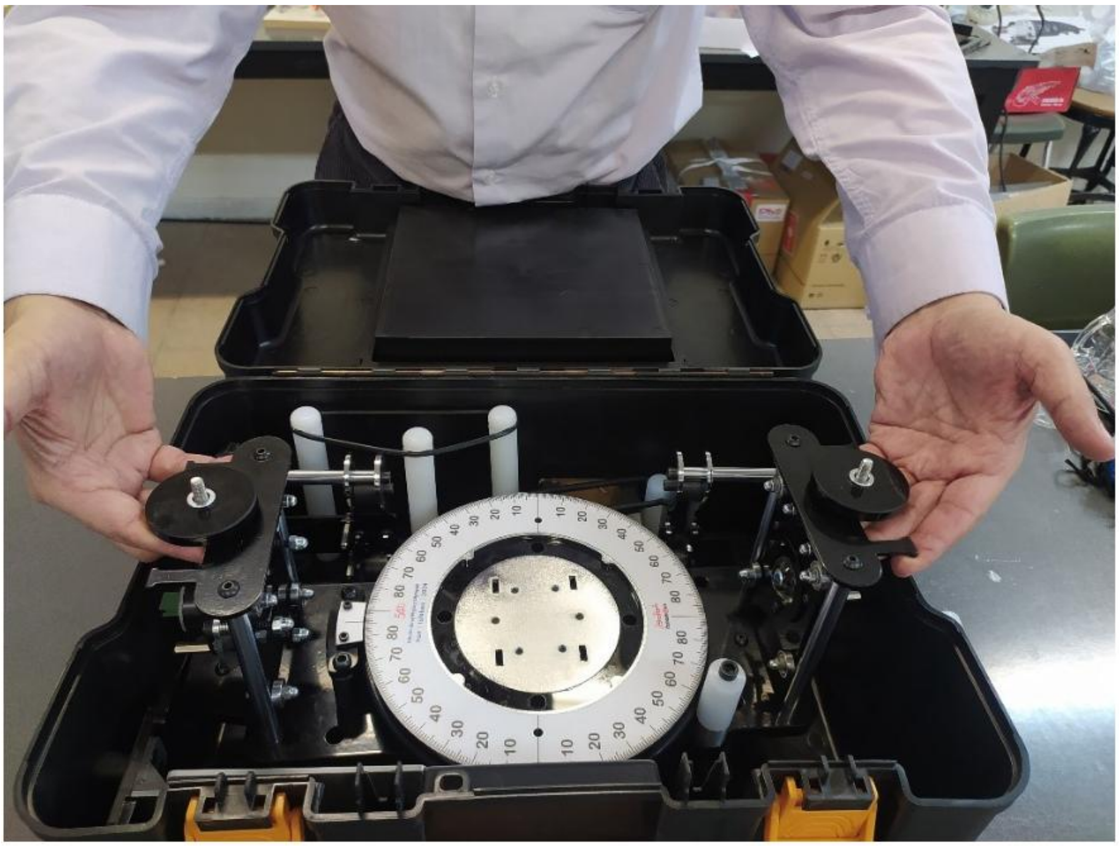


Fig. 3. Extracting the main platform from the box.

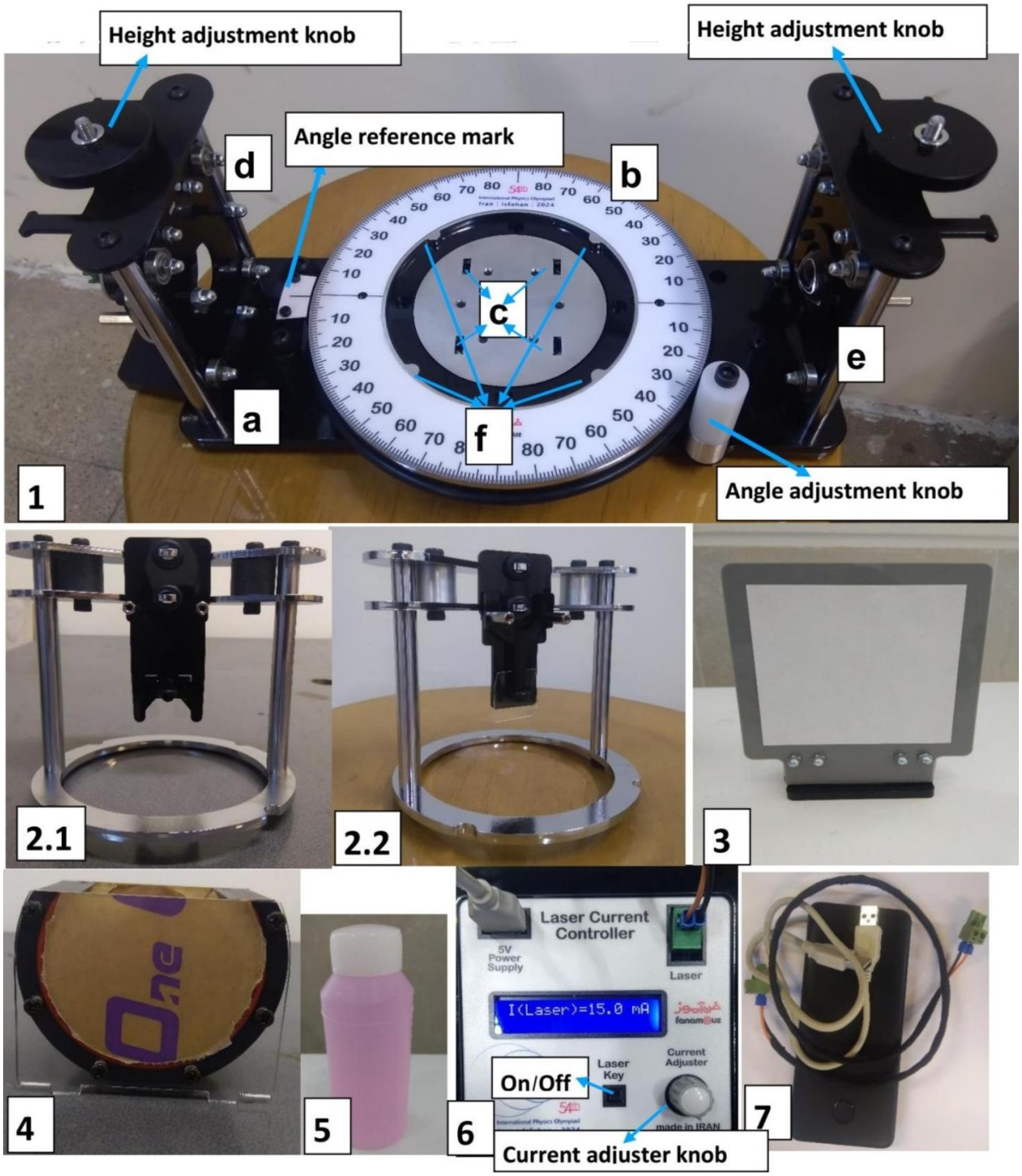


Fig. 4. The experimental setup and its component.

The setup (Details)

1. **The main platform** of the experimental setup, which consists of:
   (a). A horizontal base
   (b). A white rotating protractor: it can be rotated using the white plastic knob next to it. A reference mark on the metallic plate can be used to read the angle (see **Fig. 4.1**)
   (c). A circular plate with 4 square holes to hold the container of an unknown liquid.
   (d). A red laser and (e) convex lenses for magnifying the diffraction pattern, installed on the walls at the two sides of the platform: their height can be adjusted by turning the knobs at the top.
   (f). Four protrusions on the inner wall of the protractor to hold the glass pieces' holders.
2. **Holders S1 (2.1) and S2 (2.2**): Each holder stands on the circular metallic plate concentric with protractor and the four protrusions (**Fig. 4.1f**) keep it fixed. The S1 holder includes a black piece which holds a thin microscope slide. The lower edge of the slide is completely free and laser light can be shone onto it. The S2 Holder is quite similar to S1, the only difference being that it holds a thick microscope slide.
3. **The observation screen**: it can be placed at any distance from the setup.
4. **The unknown liquid container**: after removing the protective adhesive paper, it can be placed on the square holes in the middle of the protractor (**Fig. 4.1c**). The effect of container walls on the diffraction pattern is negligible.
5. **The pink liquid**, inside the bottle on your desk, has an unknown refractive index.
6. **The laser electronic board**: it can be turned on by connecting the laser to the board (and the board to the power bank). Use the On/Off switch on the board to turn the laser on or off. The intensity of the laser light can be adjusted by turning the current adjuster knob on the electronic board. Set the intensity of the laser to a level at which your eyes are comfortable.
7. **Power bank and electrical cables.**

Please take note of the following:

1. Do not touch the glass lens and the microscope slides at all, because your fingerprints can affect the results of your experiment, and the slides are rather thin and can easily break.
2. Do not drink the unknown liquid.
3. Do not look directly into the laser.

Theoretical Background

When a laser beam is shone at the edge of a transparent slide, a phase difference is introduced between the part that travels through the slide and the part that does not. This phase difference results in a diffraction pattern, the lines of which are parallel to the edge of the slide (see **Fig. 5**).

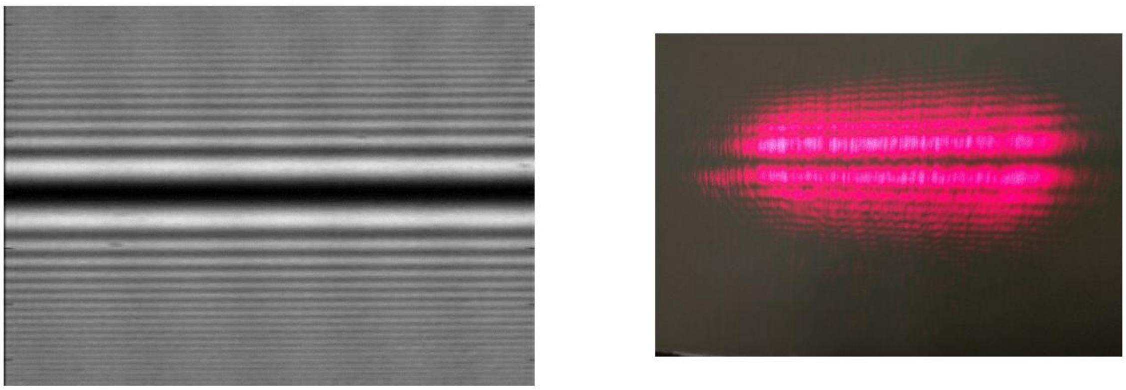


Fig. 5. A theoretical diffraction pattern (left), and the diffraction pattern observed in the lab (right)

Let us take the direction of the beam as the z-direction (see **Fig. 6**), and at first, we'll assume that the slide is in the $x - y$ plane and its horizontal edge coincides with the x-axis (i.e. the angle in **Fig. 6** is equal to zero). In this case the phase difference between the two parts of the beam clearly is:

$$\phi_0 = \frac{2\pi h}{\lambda}(n - N) \tag{1}$$

where h is the thickness of the slide, λ is the wavelength of the laser beam, N is the refractive index of the environment, and n is the refractive index of the transparent slide.

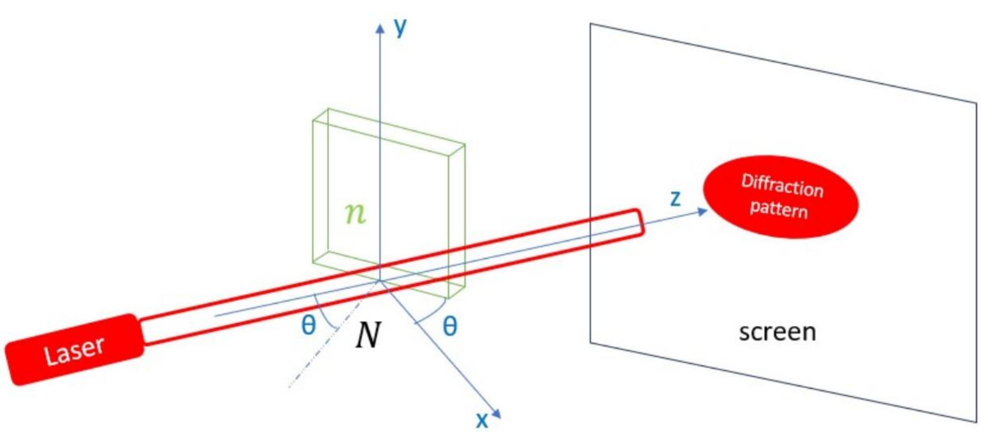
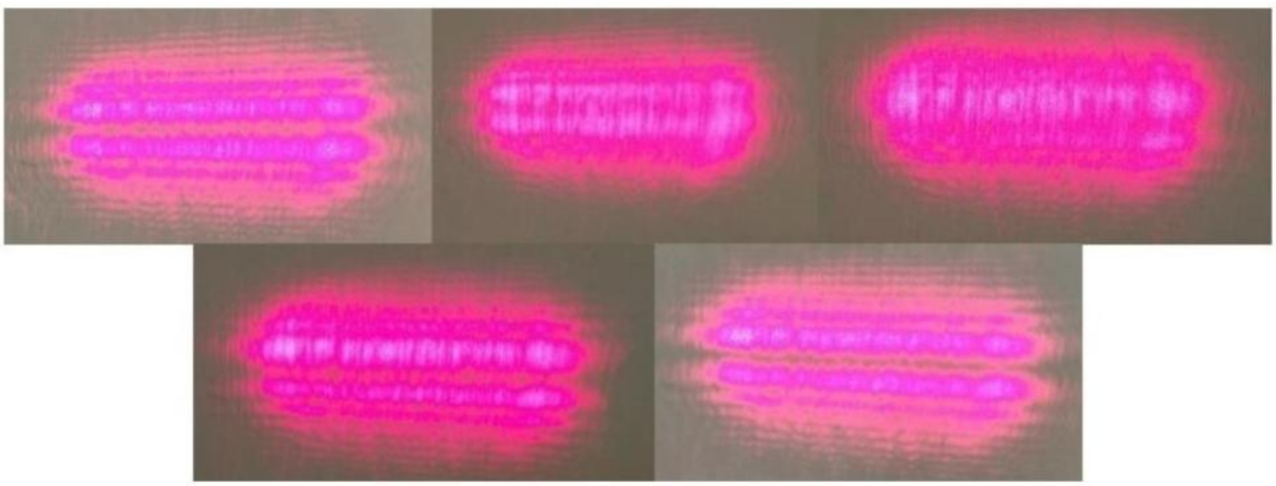


Fig. 6. Schematic of the laser beam, the slide, and the screen.

If we rotate the slide around the y-axis so that the normal to the surface of the slide makes an angle of θ with the incident beam, a simple calculation gives the following formula for the phase difference

$$\phi = \frac{2\pi h}{\lambda}\left(\sqrt{n^2 - N^2 \sin^2\theta} - N\cos\theta\right) \tag{2}$$

Hence the phase difference is a function of θ. If we continuously change this angle, the phase difference increases continuously and the shape of the pattern changes, but when the phase difference reaches 2π, the pattern reverts to its initial shape. We call this full cycle **one fringe shift**. **Fig. 7** displays the various stages of one fringe shift



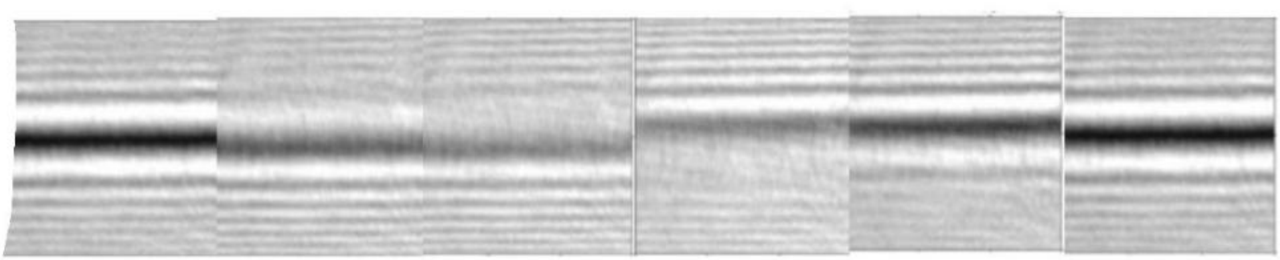


Fig. 7. Various stages of fringe shift as seen on the screen in the lab and as predicted theoretically (from left to right, each figure has phase difference equals to $\varphi, \varphi + 4\pi/9, \varphi + 6\pi/9, \varphi + 10\pi/9, \varphi + 14\pi/9, \varphi + 2\pi$).

We can start from $\theta = 0$ and gradually increase the angle. After m such fringe shifts corresponding to a rotation by $\theta = \theta_m$, we will have:

$$\phi = \frac{2\pi h}{\lambda}\left(\sqrt{n^2 - N^2 \sin^2 \theta_m} - N\cos\theta_m\right) = 2\pi m + \phi_0 \tag{3}$$

or:

$$m = \frac{h}{\lambda}\left(\sqrt{n^2 - N^2 \sin^2 \theta_m} - N\cos\theta_m\right) - \frac{\phi_0}{2\pi} \tag{4}$$

Important note:

1. You only need to calculate the uncertainty in the final results of each part (Errors need to be calculated and reported whenever the $\pm$ sign is present in the answer sheet).
2. You can use the provided calculator to find the slope and the vertical axis intercept of the curves.
3. Note:
   Regression, r, is a number between 1 and -1 showing how much the data can be fitted to a line. If $|\mathrm{r}| = 1$ it means data are completely on a line. In case we're calculating slope (B) and intercept (A) using a calculator in linear mode, we can use these formulas below in order to calculate their uncertainty:

$$\Delta B = B\sqrt{\frac{1}{n-2}\left(\frac{1}{\mathrm{r}^2} - 1\right)}$$

$$\Delta A = \Delta B\sqrt{\overline{x^2}}$$

   Which n is number of data points we've got, and $\overline{x^2}$ is average of square of x.

- You must calculate uncertainty of the slope and the intercept only by the formulas above.

Part A. Thickness of the Thin Slide (S1) (2.0 points)

For the following tasks, take the refractive index of the glass components (S1, S2) to be 1.51 and that of air to be 1.00. Take the wavelength of the red laser to be $650\ \mathrm{nm}$, and ignore any uncertainty in these values.

Turn on the laser. Place the S1 Holder on the protractor, and adjust the height of the laser such that it shines on the bottom edge of the microscope slide. Then adjust the height of the lens until you can observe the diffraction pattern on the screen (this height should almost be equal to the height of the laser beam). Note that the fringes in the diffraction pattern are horizontal. **Fig. 8** shows the experimental setup for **Part A**. Now slowly turn the protractor and observe the fringe shift.

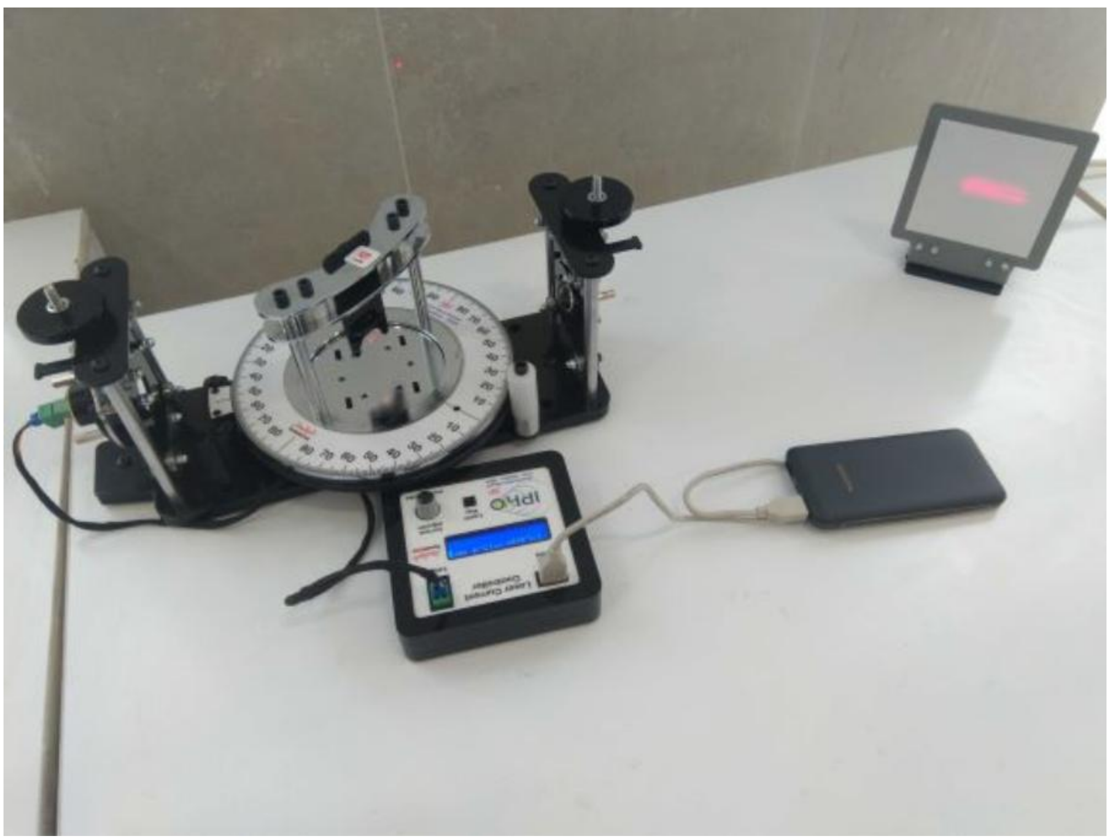


Fig. 8. Experimental setup in operation (Part A)

A.1	Starting with zero degrees, rotate the protractor and go up to 70 degrees. Watch the number of fringe shifts and write down the angle θ_m corresponding to each fringe shift number m. Take at least 25 data points and fill out the table.	0.8 pt

A.2	Draw an appropriate graph for **A.1**.	0.3 pt

A.3	Find the function slope (B) and vertical axis intercept (A).	0.1 pt

A.4	Using the slope, find the thickness of the thin slide.	0.8 pt

Part B. Thickness of the Thick Slide (S2) (1.6 points)

Go back to the setup for **Part A** using the S2 Holder instead of the S1 Holder.

B.1	Repeat the task **A.1** for θ between 0 and 20 degrees and record at least 15 data points.	0.6 pt

B.2	Assuming θ_m in **Eq. 4** is small enough, expand the relation to the order θ_m^2 and find a linear relation between the fringe shift number and θ_m^2 (assume $N = 1.00$).	0.1 pt

B.3	Draw an appropriate graph for **B.1**.	0.2 pt

B.4	Find the slope and the vertical axis intercept.	0.1 pt

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B.5	Using the slope, find the thickness of the thick slide.	0.6 pt

Part C. Finding N Using the Thick Microscope Slide (S2) (1.6 points)

Pour the unknown liquid into the container. Place the container at the center of the protractor and gently place the S2 Holder back onto the protractor in such a way that the microscope slide is immersed inside the liquid. Adjust the height of the laser and the microscope slide so that the laser beam shines at the boundary between the slide and the ambient liquid. Again, a diffraction pattern will be observed on the screen.

C.1	Repeat the task **B.1** (15 points up to 20 degrees).	0.6 pt
C.2	Repeat the task **B.2** for arbitrary N.	0.6 pt
C.3	Draw an appropriate graph for **C.1**.	0.4 pt
C.4	Find the slope and the vertical axis intercept.	0.4 pt
C.5	Find the refractive index of the unknown liquid (N).	0.2 pt

Part D. Finding N Using the Thin Microscope Slide (S1) (4.8 points)

Put S1 Holder instead of S2 inside the unknown liquid

D.1	Repeat the task **A.1** for this case (25 data points up to 70 degrees).	0.7 pt
D.2	Do a simple calculation and eliminate ϕ_0 from **Eq. 1** and **Eq. 4** to obtain a relation like $N(n-N)+uN=w$. Find u and w in terms of m, n, h, λ, and θ_m.	0.8 pt
D.3	Use your calculator to determine u and w.	1.2 pt
D.4	Draw w versus u .	0.3 pt
D.5	In the previous graph, find the linear region and calculate the slope and the vertical axis intercept of the curve.	0.2 pt
D.6	Find the refractive index, first using the slope (N_B), and then using the vertical intercept (N_A)	1.6 pt