Dark Matter (10 points)

The first formal inference for the existence of dark matter was provided by Fritz Zwicky based on his observation on the dynamics of the Coma galaxy cluster, a cluster of galaxies that consist of about a thousand of galaxies. Zwicky used the Virial theorem to estimate the mass of the galaxy cluster. In a simple sun-planet system, where the planet revolves around the sun in a circular orbit, the Virial theorem states that the kinetic energy of the planet is exactly related to its gravitational potential energy. While in a general case for a system of many particles bounded by some interaction, the Virial theorem, relates the time averaged total kinetic energy with its time averaged total potential energy.

In 1933, based on his observation on the velocity of the galaxies near the edge of the Coma galaxy cluster, Zwicky estimated that the cluster has more mass than what was visually observed (i.e. the galaxies). The gravitational attraction from the observable matter (the galaxies) was too small to account for the velocities of the galaxies. Thus there must be some hidden masses that account for such a large velocity.

That hidden mass is the dark matter mass. In what follows, assume that the mass of each galaxy is the sum of its visible mass and the mass of the dark matter which moves together with that galaxy, and dark matter interacts with visible matter only gravitationally.

Part A. Cluster of Galaxies (5.2 points)

Consider a cluster of galaxies consist of a large number N of galaxies and dark matter that are distributed homogeneously in a sphere of radius R with the total mass (galaxies and dark matters) of the cluster M. Assume that the average total mass (visible and dark matter) of a galaxy is m.

A.1	Assuming a continuous distribution of matter in the cluster, find the total gravitational potential energy of the cluster, in terms of M and R.	1.0 pt

Due to cosmological expansion, any distant object will be moving away from an observer on Earth with a speed that depends on the distance from the observer to the object. A certain Lyman (a hydrogen emission spectral line) frequency from a type IA supernova on the i-th galaxy in the galaxy cluster is observed to be f_i, with $i = 1, \dots, N$, while the same corresponding Lyman frequency on Earth is f_0.

A.2	Determine the average speed V_{cr} of the whole galaxy cluster moving away from the Earth in terms of f_i (with $i = 1, \ldots, N$.), f_0 and N. Note that a galaxy speed is very small compared to the speed of light c.	0.5 pt

A.3	By assuming that the galaxies velocities with respect to the center of the cluster are isotropic (the same in all direction), determine the root-mean-square speed v_{rms} of the galaxies with respect to the center of the cluster in terms of N, f_i (with $i = 1, \ldots, N$.), and f_0. From this result determine the mean kinetic energy of a galaxy with respect to the center of the cluster in terms of v_{rms} and m.	1.5 pt

To find the total mass of the cluster, one can use Virial theorem. The theorem states that for a system of particles bounded by their conservative force,

$$\langle K \rangle_t = -\gamma \langle U \rangle_t, \tag{1}$$

where is the time averaged total kinetic energy, is the time averaged total potential energy, and is a constant. This theorem can be derived by assuming that for a system of particles bounded by its own interaction, the magnitudes of the position and the momentum of each particle are finite, and thus the following virial quantity

$$\Gamma = \sum_i \vec{p}_i \cdot \vec{r}_i \tag{2}$$

is finite.

A.4	Using the fact that the time average over a long period of time of $\frac{d\Gamma}{dt}$ vanishes, $\left\langle \frac{d\Gamma}{dt} \right\rangle_t = 0$, determine γ in the Virial theorem above for the case of gravitational interaction. (Hint: Try to do the problem with the summation in Γ for a small finite number of galaxies).	1.7 pt

A.5	From the previous results determine the total dark matter mass of the cluster in terms of N, m_g, R, and v_{rms}, where m_g is the average total visible mass of the galaxy. Note that the root-mean-square speed of the dark matter is the same as that of the galaxies.	0.5 pt

Part B. Dark Matter in a Galaxy (2.8 points)

Dark matter also exists inside and around a galaxy. Consider a spherical galaxy with a visible edge radius R_g (an approximate outermost distance where a large number of stars is still visible, but note that a very small number of stars may still be distributed in the region beyond R_g). Assume that the stars in the galaxy are point particles with an average mass m_S. The stars in the galaxy, distributed homogenously with a number density n, are assumed to move in circular orbits.

B.1	If the galaxy consists only of stars, find the speed of a star $v(r)$ as a function of its distance to the center of the galaxy r and sketch $v(r)$ for $r < R_g$ and $r > R_g$.	0.8 pt

The existence of dark matter can be inferred from the galaxy rotation curve, which is a plot of $v(r)$ obtained from observations. The figure below shows a common pattern of the galaxy rotation curve. You may assume simply that $v(r)$ is a linear function for $r < R_g$ and a constant for $r > R_g$.

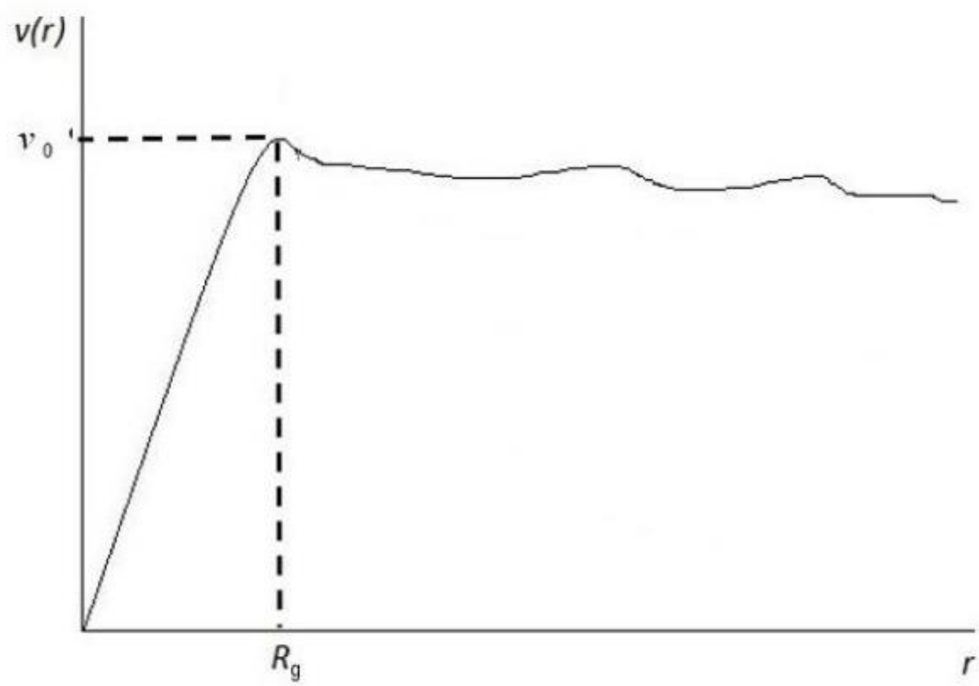

Fig. 1. Plot of galaxy rotation curve in a galaxy.

B.2	Find the total mass m_R of that part of the galaxy which lies within the sphere of radius R_g in terms of v_0 and R_g.	0.5 pt

The discrepancy between the **Fig. 1** and the plot obtained in **B.1** indicates the existence of dark matter.

B.3	Determine the dark matter mass density as a function of r, R_g, v_0, n, and m_S for $r < R_g$ and $r > R_g$.	1.5 pt

Part C. Interstellar Gas and Dark Matter (2.0 points)

Now consider a young galaxy whose mass is dominated by interstellar gas and dark matters (neglect the mass of the stars). The interstellar gas is assumed to consist of identical particles of mass m_P. The number density $n(r)$ and temperature $T(r)$ of the gas depends on the distance from the center of the galaxy r. Although many physical processes happen in the gas, we can assume that the gas is in hydrostatic equilibrium due to its pressure and the galaxy gravitational interaction.

C.1	Find the pressure gradient of the gas $\frac{dP}{dr}$, in terms of $m'(r)$, r, and $n(r)$. Here, $m'(r)$ is the total mass of gas and dark matters within a sphere of radius r from the galaxy center.	0.5 pt

C.2	Assuming the interstellar gas as an ideal gas, find $m'(r)$ in terms of $n(r)$, $T(r)$ and their derivatives with respect to r.	0.5 pt

Next, for simplicity assume that the gas is in isothermal distribution at temperature T_0 and the interstellar gas number density is given by

$$n(r) = \frac{\alpha}{r(\beta + r)^2} \tag{3}$$

where α and β are some constants

C.3	Find the dark matter density as a function of r inside the galaxy	1.0 pt

Earthquake, Volcano, and Tsunami (10 points)

Indonesia is the supermarket of natural disasters. Almost all of the natural disasters have occurred in Indonesia, such as volcano eruptions, earthquakes, and tsunami.

Part A. Merapi Volcano Eruption (1.3 points)

Merapi volcano in Yogyakarta is one of the most active volcano in Java. Pyroclastic flows are well-known eruption characteristics of the volcano. The pyroclastic flow is a hot mixture of gas and rock which travels away from a volcano. In October 26th , 2010, Merapi showed his explosive character by producing an ash plume that reached 12 km altitude (**Fig.1**) and pyroclastic currents displacing more that 20,000 people around the volcano.

Let us look at the causes of the largest eruption of Merapi in 2010. It is known by geophysicists that the influence of the external water into the magma plays an important role to the explosive behavior of volcanic eruptions (hydro-magmatic eruptions). Let's assume that we dealt with a volcano as a system that consists of mixture of magmatic particles and water. The volcano vents structures and atmosphere are being boundary of the system.

Fig. 1. Pyroclastic cloud during Merapi eruption,

(Courtesy of Volcanological Office of Yogyakarta, BPPTKG)

The explosive eruption is considered to be happening in two stages, **(1)** an instantaneous magma-water interaction, and **(2)** a system expansion. In the first stage, a mass of magma (m_m) at an absolute temperature (T_m) mixes with a mass of external water (m_w) at an absolute temperature (T_w).

The thermal equilibrium is reached almost instantly. This interaction can be perceived as a nearly-constant volume process. Latent heat of evaporation of water and latent heat of melting of magma can be neglected.

A.1	Find the equilibrium temperature at the first stage in terms of the masses and heat capacity per unit mass of water c_{Vw} and magma c_{Vm}.	0.5 pt

A.2	Determine its equilibrium pressure at the first stage by assuming that the mixture can be modeled as ideal gas. Assume that the volume per unit mole of the mixture is v_e.	0.3 pt

The system expansion (the second stage) can occur through several possibilities, one of which is thermal detonation. Although such process is quite complicated, we can empirically measure the relative velocity of the erupted mixture. The velocity of gas during the eruption depends on the pressure p, the total mass m, and the volume V of the mixture in the conduit of the volcano

A.3	Express the velocity of the gas during the eruption in terms of p, m, and V up to a proportional constant κ.	0.5 pt

The observed pressure is around the order of 100 MPa. This can make the eruption (relative) velocity be as large as ballistic velocity.

Part B. The Yogyakarta Earthquake (5.7 points)

The 2006 Yogyakarta earthquake of magnitude $M_w = 6.4$, which destroyed many buildings in the Bantul and Yogyakarta area, occurred at 05:54:00.00 local time or 22:54:00.00 UTC. The earthquake was caused by a sudden displacement of the Opak fault segment (see **Fig. 2**). The hypocenter was located 15 km below the surface.

The seismic wave that propagates on the Earth crust can be recorded using seismometer. The diagram from seismometer is called seismogram (**Fig. 2** and **3**, Lower graph). The seismograms represent the vertical ground velocity as a function of time recorded by the seismic station at Gamping Station Yogyakarta (YOGI) (**Fig. 2**) and Denpasar, Bali (DNP) (**Fig. 3**).

In general, seismic wave consists of three wave types: the longitudinal or primary (P-wave), the transversal or secondary (S-wave), and the surface wave. The P-wave and S-wave travels in the subsurface while the surface wave travels along the Earth's surface.

Seismic waves traveling though subsurface to the stations can be divided into those that propagate in a straight line, those that are reflected by a layer's boundary, and those that are refracted into the next layer. The longitudinal wave or the primary wave has the highest velocity, while the surface wave has the lower velocity, around 60% of the P-wave.

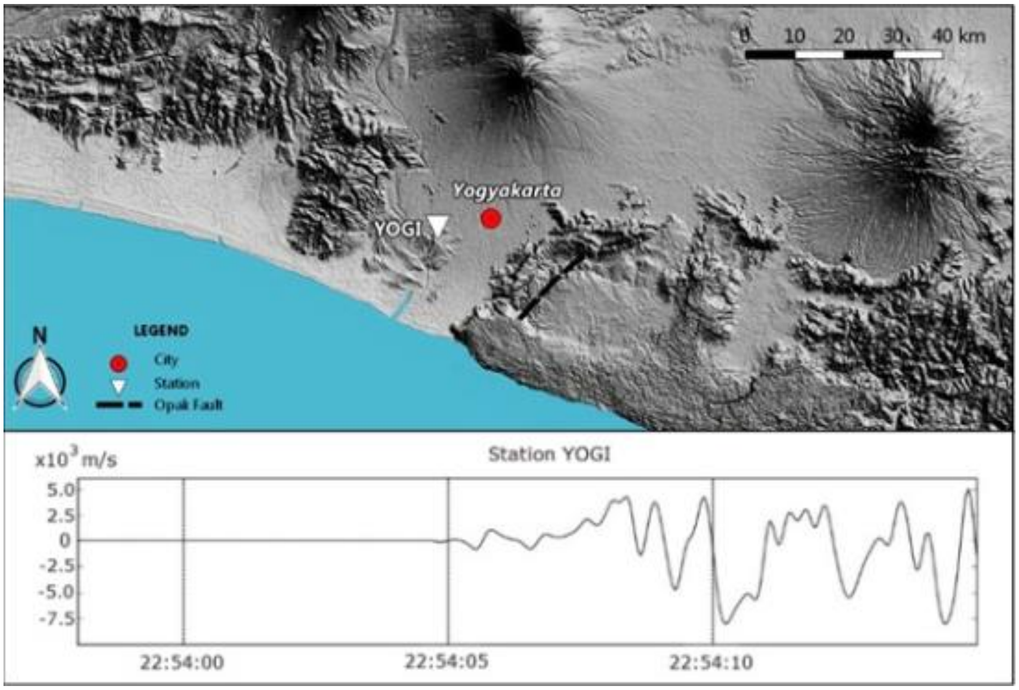

Fig. 2. The maps location of YOGI.

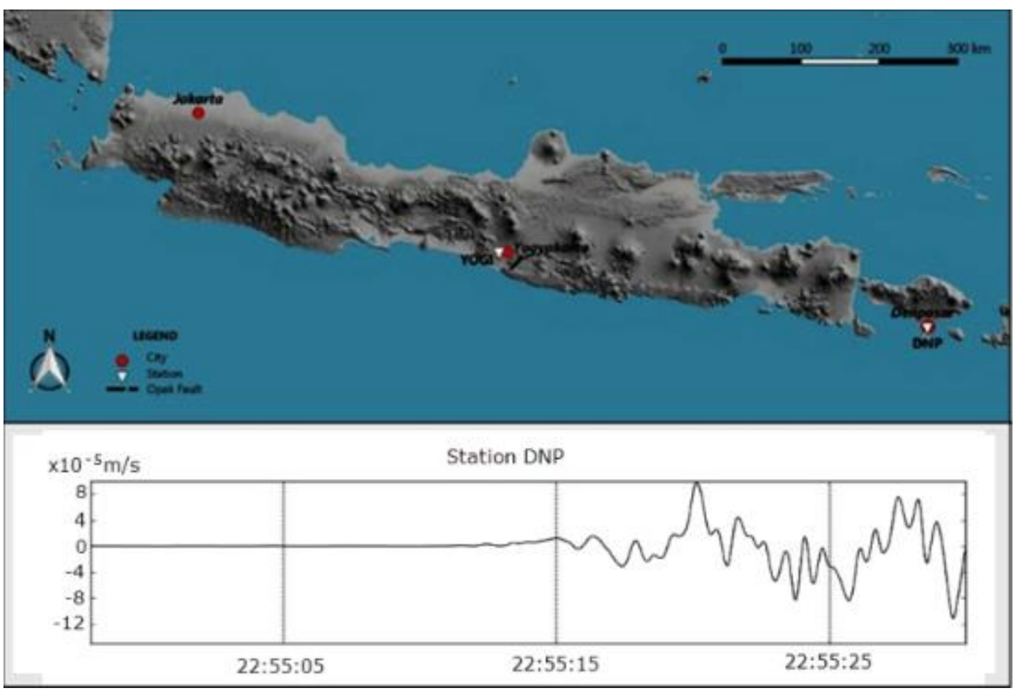

Fig. 3. The maps location of DNP

The distance between the epicenter (the projection of the hypocenter on the Earth surface) and the YOGI and DNP stations respectively are 22.5 km and 500 km. The depth of the Earth crust later in Java, Indonesia, is 30 km. Beneath the Earth's crust is the Earth's mantle layer.

Just like other wave phenomena, seismic wave also satisfies the Snell's law. The seismic wave can also be reflected by the mantle layer. In this problem we assume that the Earth's curvature is neglected.

B.1	**Fig. 2** shows the seismogram at the YOGI station. Assuming that the seismogram reads zero until the seismic wave travels to the station, use the data to find the velocity of the -wave in the Earth's crust.	0.5 pt

B.2	From your result in **B.1**, find the travel time of the direct P-wave and the reflected wave due to the Yogyakarta earthquake that arrived at the DNP station in Denpasar.	0.6 pt

By assuming that the Earth is composed of only two layers: the crust and the mantle, the primary wave propagates in the crust and in the mantle with different constant velocities. The velocity in the mantle is faster than in the crust.

Note that the P-wave which refracts into the mantle at the right angle $(90°)$, and is partly refracted back into the crust along its entire path of propagation along the crust-mantle boundary, and the wave propagating through the crust-mantle boundary arrives at the station faster than the direct and the reflected P-wave

B.3	Using the data in **Fig. 3**, find the velocity of the P-wave in the mantle	1.2 pt

For a more realistic Earth structure, the crust can be divided into a number of thin layers so that the velocity of the seismic wave is a function of the depth z according to $v(z) = v_0 + az$ where a is a constant and the hypocenter is approximated to lie on the surface. In this model, the wave ray is curving.

B.4	Let us define the ray parameter $p = \frac{\sin\theta(z)}{v(z)}$, where $\theta(z)$ is the angle between the ray and the normal. Suppose that a seismic wave arrives to the station with ray parameter p; express the distance to the hypocenter in terms of p, v_0, and a. Assume that the hypocenter is very close to the ground surface.	1.4 pt

B.5	Find the travel time T from the hypocenter to any station, in the form of an integral over z.	1.0 pt

The Earth consists of a stack of homogeneous layers with the seismic velocity v_i and thickness of δz_i for each layer

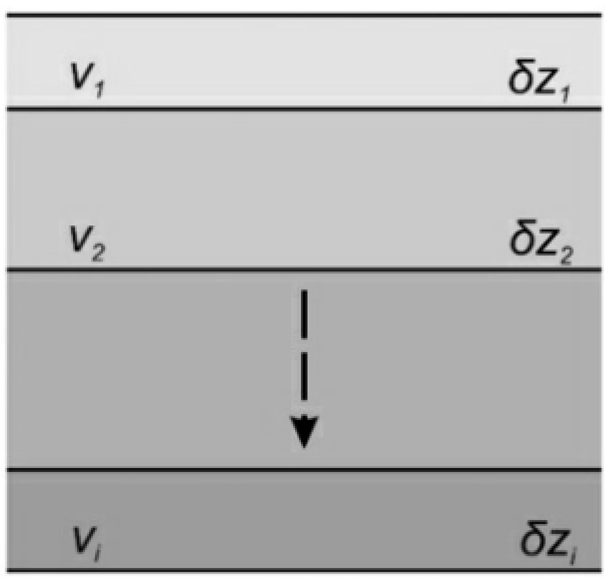

Fig. 4. A simplified model of Earth's

B.6	From the result of **B.5**, approximate the travel time (T) from the hypocenter to DNP station by assuming that the crust consists of only three discretized layers, $(i = 1, 2, 3)$, characterized by $v_1 = 6.65$ km/sec, $v_2 = 6.97$ km/sec, $v_3 = 6.99$ km/sec, $p = 0.143$ sec/km, $\delta z_1 = 6.0$ km, $\delta z_2 = 9.0$ km, $\delta z_3 = 15$ km.	1.0 pt

Part C. Java Tsunami (3.0 points)

The 2006 Pangandaran earthquake and tsunami occurred on July 17 at 15:19:27 local time off the coast of west and central Java. During the earthquake where the epicenter fault is on the ocean floor, the fault may be displaced, producing a remarkably large water wave called tsunami. In other words, a tsunami is a shallow-water wave which is initiated by a tiny amplitude, but with an extremely large wavelength.

Consider a sudden fault causing a lifting of some ocean floor as shown in **Fig. 5**. Assume that the energy of the earthquake is transformed into the potential energy of this raised ocean water. For a simple model we approximate that the raised water has a geometry of a box with its area of $\lambda L/2$ (where $L \gg \lambda$) and height of h, the ocean depth at this point is d.

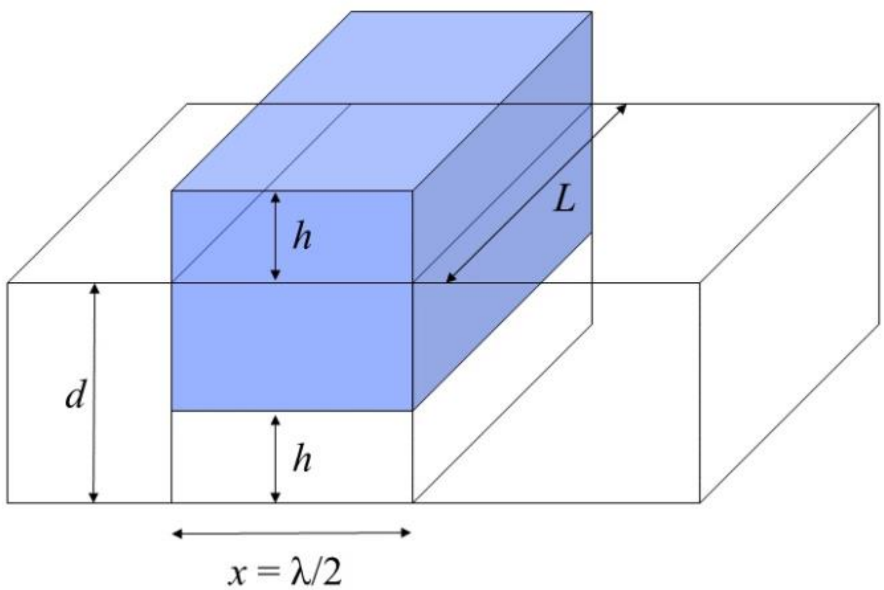

Fig. 5. Illustration for the tsunami wave, d is the depth of the ocean.

C.1	Find the potential energy stored in the raised ocean water due to the earthquake with respect to the ocean surface. Assume that the density of seawater is ρ.	0.5 pt

C.2	Find the speed of the tsunami wave up to a dimensionless factor.	1.2 pt

C.3	Using energy argument, determine the amplitude of the tsunami wave as a function of the depth, assuming that the depth varies slowly and also knowing that at a depth of d_0 the amplitude is A_0	1.3 pt

Cosmic Inflation (10 points)

Due to the relative movement of galaxies observed from the Earth, the wavelength of visible spectrum of a particular galaxy differs from its original wavelength, which is known as the electromagnetic Doppler effect. One expects, for a collection of galaxies, to a random distribution of wavelength shifts: some positive (redshifted) and some negative (blueshifted).

However, observation shows that all, except for a nearby group of galaxies, are redshifted. This must be true even if the observation takes place on a different point in the universe. As a conclusion, our universe must be expanding.

On the other hand, local irregularity of the universe can be neglected on scales of more than 100 Mpc, in which 1 pc=3.26 lightyears. Averaged over large scales, the clumpy distribution of galaxies becomes more and more isotropic (independent of direction) and homogenous (independent of position). Therefore we can assume the universe as a matter having a uniform mass density ρ and is expanding.

Part A. Expansion of the Universe (5.0 points)

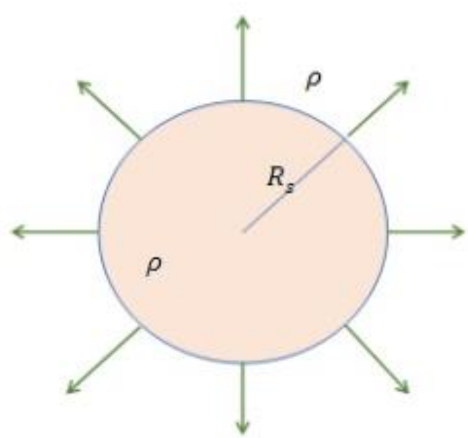

Fig. 1. The universe modeled as a uniform sphere.

For a simple model of our universe, let us consider an expanding uniform-density sphere embedded in a medium of much larger sphere with the same density. Let's say at some time, the radius of the smaller sphere is R_S. To express the expansion of the sphere, the time dependence of the radius $R(t)$ can be expressed by a scale factor $a(t)$, that is $R(t) = a(t)R_S$.

Using Newton's law of gravitation to evaluate the velocity of a mass element on the sphere boundary according to the model of our universe, one can obtain the 1st Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = A_1\rho(t) - \frac{kc^2}{R_S^2 a^2(t)} \quad (1)$$

where k is a dimensionless constant, and c is the speed of light.

A.1	Determine the constant A_1 in **Eq. (1)**	1.3 pt

The discussion so far is non-relativistic. But in fact, it can be extended to a relativistic system by reinterpreting $\rho(t)c^2$ as the total energy density (excluding the gravitational potential energy). In this relativistic system one obtains the conservation equation:

$$\dot{\rho} + A_2\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0 \quad (2)$$

using the 1st law of thermodynamics of an adiabatic system, where p denotes the pressure on the sphere.

A.2	Determine the constant A_2 in **Eq. (2)**	0.9 pt

To solve **Eqs. (1)** and **(2)**, one should assume a relation $p = p(\rho)$, such as $p(t)/c^2 = w\rho(t)$, where w is a constant. There is also a factor $H = \dot{a}/a$ called the Hubble parameter. The present values of parameters are usually symbolized by the subscript 0, such as t_0, ρ_0, H_0, a_0, and so on. For simplicity, we take $a_0 = 1$.

The universe is believed to start from a big explosion called the Big-Bang, that produces radiation of relativistic particles. During its expansion, the universe cools down and the particles inside becomes non-relativistic. However, the recent observation clarify that the present universe is dominated by cosmological constant energy density. For the case of photon, as the universe is expanding, the photon's wavelength expands proportionally to the scale factor.

A.3	For each of the following three cases, determine the resulting value of w: (i) A universe filled only with radiation (i.e. photon energy) (ii) A universe filled only with non-relativistic matter (iii) A universe with constant energy density	1.2 pt

A.4	In the case of $k = 0$, find for each case of (i) to (iii) being mentioned in **A.3**. Use the initial condition $a(t = 0) = 0$ for cases (i) and (ii), and use the condition $a_0 = 1$ for case (iii).	1.2 pt

The constant k in **Eq. (1)** refers to the classification of spatial geometry of the universe. Its value can be $k = +1$ for positive-curvature universe (closed), $k = 0$ for flat universe (infinite), and $k = -1$ for negative-curvature universe (open, infinite). Let's define a density ratio $\Omega = \rho/\rho_c$, where $\rho_c c^2 = H^2/A_1$ is the critical energy density. Note that A_1 is obtained from **A.1**. We will now choose an appropriate $R_0 = a_0 R_s$ for the next two tasks such that the value of k can be normalized.

A.5	Express k in **Eq. (1)** in terms of Ω, H, a, and R_0.	0.1 pt

A.6	Find a range for Ω that corresponds to each value of $k = +1$, $k = 0$, and $k = -1$.	0.3 pt

Part B. Motivation to Introduce Inflation Phase and its General Conditions (1.9 points)

The observation of cosmic microwave background radiation (CMB) suggests that our present universe is **approximately flat**. The problem is that if this is true then the present universe should start from exactly flat universe, otherwise any deviation from the flatness will eventually grow over time and spoil the present flatness.

B.1	Using the flat universe solution obtained in **A.3**, find $\Omega(t) - 1$ as a function of time for the **approximately flat** universe of general k when it is dominated by radiation or non-relativistic matter.	0.5 pt

To solve the problem, at some early time in its history, the universe should undergo a constant energy density domination period which leads to an exponential expansion, so called inflation period.

B.2	For this constant energy density domination period in the **approximately flat** universe, find $\Omega(t) - 1$ as a function of time. Assume $\Omega(t) - 1 \ll 1$.	0.3 pt

B.3	Show that the inflation period implies several following consequences: Negative pressure $(p < 0)$, accelerated expansion $(\ddot{a} > 0)$, and decreasing Hubble radius $\left(\frac{d\left(aH^{-1}\right)}{dt} < 0\right)$.	0.9 pt

B.4	Show that the condition of decreasing Hubble radius can be expressed in terms of the parameter $\varepsilon = -\dot{H}/H^2$ as $\varepsilon < 1$.	0.2 pt

Inflation occurs as long as $\varepsilon < 1$ and then ends when $\varepsilon = 1$. We can define the e-folding number N such that $dN = d\ln a = Hdt$ and $N = 0$ at the end of the inflation.

Part C. Inflation Generated by Homogenously Distributed Matter (1.7 points)

An example of a simple physical system that can generate a period of inflation is a universe dominated by a homogeneously distributed exotic matter. This kind of exotic matter is called inflaton, and can be characterized by a function $\phi(t)$.

The dynamical equation of the matter can be expressed as

$$\ddot{\phi} + 3H\,\dot{\phi} = -V', \tag{3}$$

where $V = V(\phi)$ is a potential function and $V' = \frac{\partial V}{\partial \phi}$. The Hubble parameter satisfies

$$H^2 = \frac{1}{3M_{pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right] \tag{4}$$

with a constant M_{pl} called the reduced Planck mass. Inflation phase occurs during the domination of potential energy V over kinetic energy $\dot{\phi}^2/2$ for a sufficient time such that the $\ddot{\phi}$ term in Eq. (3) can be disregarded. This condition is called the slow-roll approximation.

The quantities ε and $\eta_V = \delta + \varepsilon$, where $\delta = -\ddot{\phi}/(H\dot{\phi})$, are called “slow-roll” parameters.

C.1	Estimate the parameter ε, η_V, and $dN/d\phi$ in terms of the potential $V(\phi)$ and its first and second derivatives (V' and V'').	1.7 pt

Part D. Inflation with a Simple Potential (1.4 points)

Predictions of any inflation model should be compared to observational constraints from CMB as follows: $n_s = 0.968 \pm 0.006$ and $r < 0.12$, where $r = 16\varepsilon$ and $n_s = 1 + 2\eta_V - 6\varepsilon$ are evaluated at $\phi = \phi_{start}$ for an inflation model being generated by a dominant matter. Assume that the potential of the matter takes a simple form $V(\phi) = \Lambda^4 \left(\frac{\phi}{M_{pl}}\right)^n$ where n is any integer and Λ is a constant.

D.1	Calculate ϕ_{end} at the end of the inflation period.	0.5 pt

D.2	Express r and n_s in terms of the e-folding number N and the integer n. Estimate the value of that is closest to the observational values r and n_s. Take $N = 60$ in your calculations.	0.9 pt

Measuring the Radius of a Star (10 points)

Measuring the radii of stars is a difficult task due to their great distances. Atmospheric inhomogeneities blur the image, making it difficult to determine their true size. One possible method for measuring stellar radii is using an interferometer. This method was proposed by Michelson and Pease in 1921 to measure the diameter of Betelgeuse.

Fig. 1 below shows a diagram of the interferometer used to measure the star's radius.

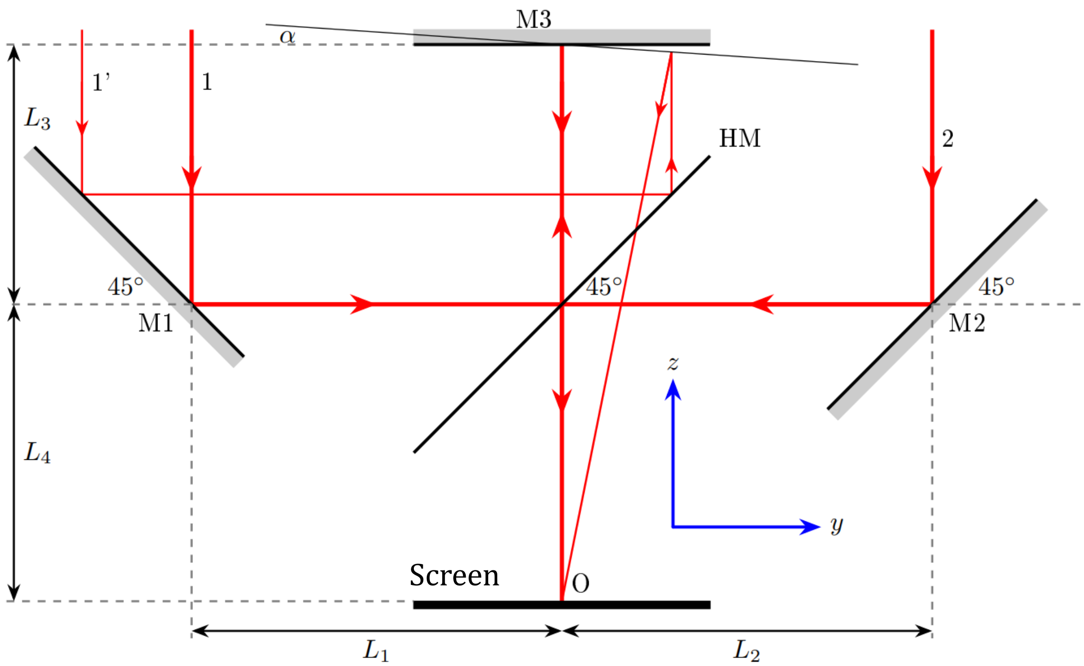

Fig. 1. The path of the light rays coming from a

Reflecting from mirror **M1**, beam **1** hits beamsplitter **HM** which reflects 50% of light intensity and transmits the remaining 50%. The reflected beam hits mirror **M3**, which is reflected, and passes through HM, and then hits the screen. At the same time, beam **2** is reflected from mirror **M2**. Reflected from **HM**, it also hits the screen. Consider that mirrors **M1**, **M2**, and **M3** reflect all of the light incident on them, and adds a phase of π radians to the light. The wavelength of the light used is λ.

Part A. Rays Parallel to the z-axis (1.6 points)

A.1	Let's consider the case when rays **1** and **2** are parallel to the z-axis, and mirror M3 parallel to the xy-plane. Find the optical path difference Δ_{12} between rays **1** and **2** if they both fall on the screen at a point O.	0.2 pt

A.2	Let us consider the case when mirror M3 is tilted at an angle α from the xy-plane by rotating around the x-axis (see **Fig. 1**). If ray **1'** falls on the screen at point O, find the optical path difference $\Delta_{1'2}$ between rays **1'** and **2**. Show that, to a first approximation, the path difference does not depend on and simplify your answer when $\alpha \ll 1$.	1.4 pt

Part B. Rays at an Angle θ in the yz-plane (4.7 points)

Now let the rays fall on the interferometer at an angle $\theta \ll 1$ to the z-axis in the yz-plane, as shown in **Fig. 2** below. As in A.2, the mirror **M3** is tilted by an angle α.

Note: In all further points of the problem, work only in the linear approximation when calculating optical paths, i.e., neglect terms containing factors α^2, θ^2, $\alpha\theta$ etc.

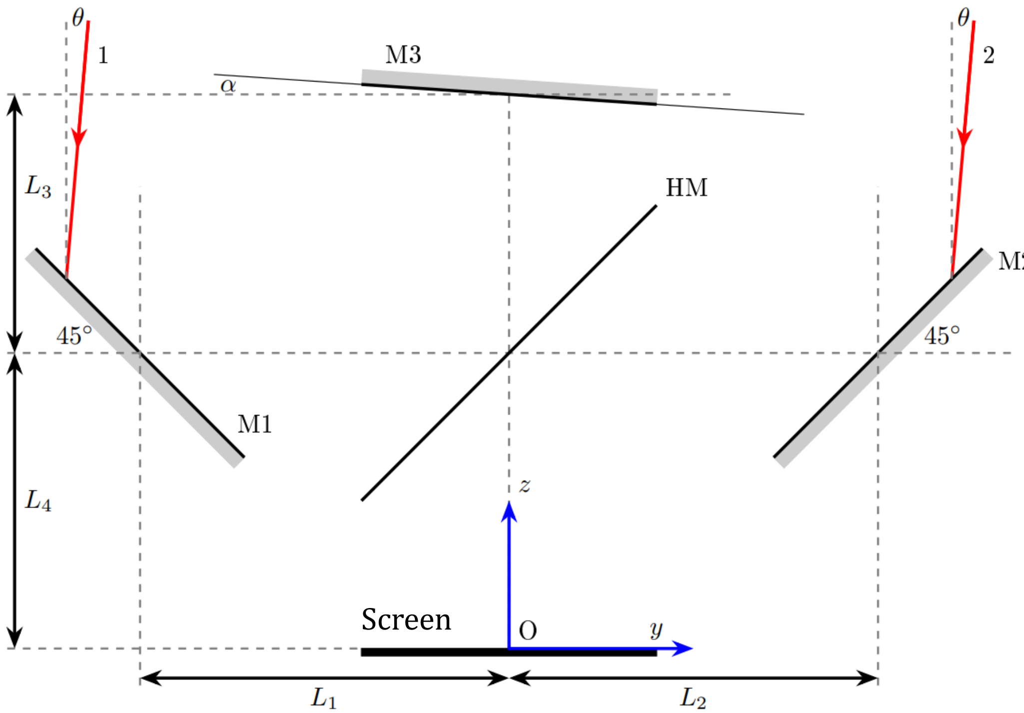

Fig. 2. Rays incident at an angle θ in the yz-plane and falling on the screen at point O.

B.1	Let rays **1** and **2** be incident on fixed points of mirrors **M1** and **M2**, respectively. Show on your working sheets that, to the first approximation, the optical paths of rays **1** and **2** from the reflection points of mirrors **M1** and **M2**, respectively, to the screen are independent of the angles θ and α.	0.8 pt

B.2	Let the angles α and θ be fixed, and the positions of the points of incidence of rays **1** and **2** on mirrors **M1** and **M2**, respectively, can change. Show in the working sheets that the optical paths of rays **1** and **2** from the dotted line corresponding to the coordinate (meaning the points of intersection of the dotted line with the trajectories of rays **1** and **2** before falling on mirrors **M1** and **M2**, respectively), to the screen, in the first approximation, do not depend on the angles θ and α.	0.8 pt

If rays **1** and **2** fall on the screen at point O, the phase difference between them can be written as:

$$\delta_{12} = \frac{2\pi}{\lambda} b\theta + \delta_O.$$

B.3	Express b and δ_O through λ, L_1, L_2, L_3, and L_4.	1.0 pt

If ray **1'**, reflected from mirror **M1**, and ray **2'**, reflected from mirror **M2**, fall at the same point on the screen with coordinate y, the phase difference between them can be written as:

$$\delta_{1'2'} = \frac{2\pi}{\lambda} b\theta + \delta_y.$$

B.4	Show in the working sheets that $$\delta_y = \delta_O - \frac{2\pi}{\lambda} \cdot 2\alpha y$$	0.6 pt

Note: Further in all points of the problem you can use the statements in points **B.1**-**B.4**, even if they are not proven.

B.5	Thus, an interference pattern will appear on the screen. Find the distance y_1 from point O to the nearest interference maximum. Find also the distance Δy_m between two adjacent interference maxima. Provide numerical answers in the case $L_1 = 0.50\ \mathrm{m}$, $L_2 = 1.50\ \mathrm{m}$, $L_3 = 0.50\ \mathrm{m}$, $L_4 = 1.00\ \mathrm{m}$, $\alpha = 10^{-4}$, $\theta = 10^{-5}$, $\lambda = 500\ \mathrm{nm}$.	0.6 pt

B.6	Find how the intensity $I(y)$ on the screen depends on the coordinates y, if its maximum value is equal to I_0, substituting the numerical values from **B.5**. Consider the effect of the beam splitter on the intensity of the transmitted light.	0.9 pt

Part C. Completely Arbitrary Rays (1.0 points)

Now let the rays **1”** and **2”** fall on the interferometer at an angle $\theta_x \ll 1$ to the yz-plane and at an angle θ_y to the xz-plane (see **Fig. 3**).

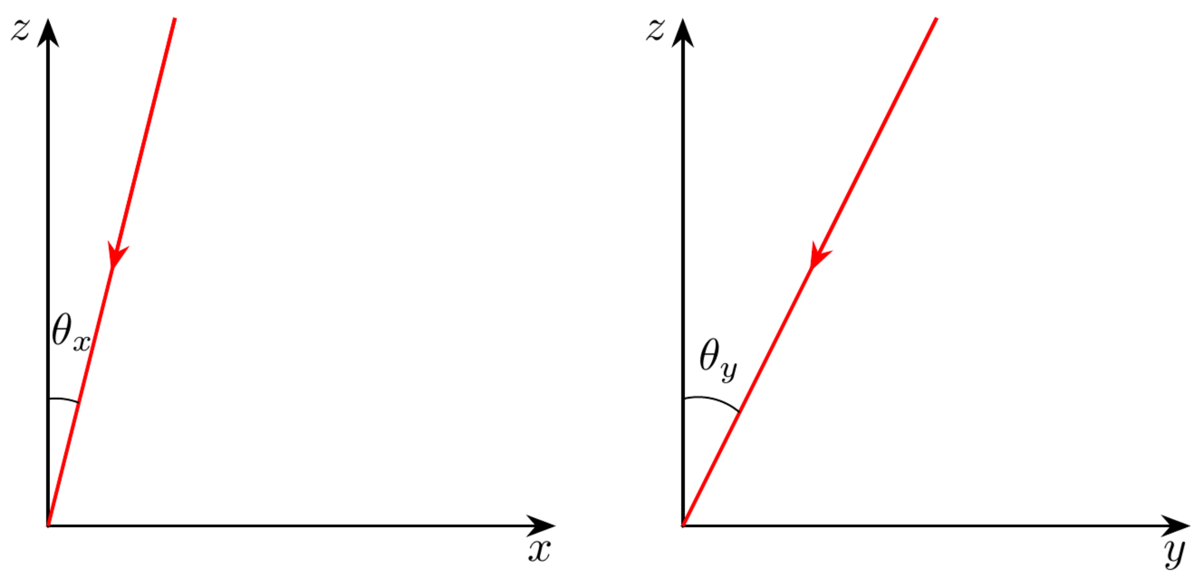

Fig. 3. The orientation of the rays **1”** or **2”**

C.1	Express the wavevector $\vec{k}$ of the incident rays through θ_x, θ_y, the modulus of the wavevector $k = 2\pi/\lambda$, and unit vectors of the coordinate axes.	0.4 pt

Let ray **1**, reflected from mirror **M1**, and ray **2**, reflected from mirror **M2**, fall at the same point on the screen on the y-axis at a distance y from the point O.

C.2	Express the phase difference between beams **1"** and **2"** through λ, L_1, L_2, L_3, L_4, θ_x, θ_y, α, and y.	0.6 pt

Part D. Measuring the Radius (2.7 points)

Let us consider a star with an angular radius θ_R, which we need to determine. For simplicity, we will assume that the star radiates as a uniform disk, i.e., the intensity of light coming from solid angle $\mathrm{d}\Omega = \mathrm{d}\theta_x \mathrm{d}\theta_y$ is equal to $\mathcal{J} d\Omega$, where $\mathcal{J}$ is a constant value in any direction inside the star's disk.

Let the interferometer (or more precisely, the z-axis) be directed exactly to the center of the star. At $\alpha > 0$, an interference pattern will be observed on the screen. Let us denote the minimum and maximum intensities on the screen along the y-axis as $I_{y(min)}$ and $I_{y(max)}$ respectively. Let us introduce the *visibility* $\mathcal{V}$ of the interference pattern as:

$$\mathcal{V} = \frac{I_{y(max)} - I_{y(min)}}{I_{y(max)} + I_{y(min)}}$$

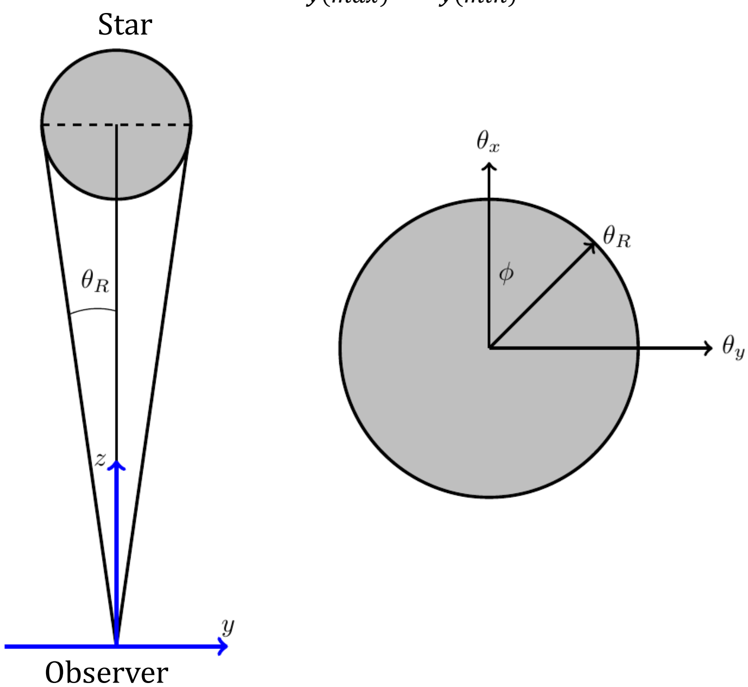

Fig. 4. Left: Side view of the star and the observer;

Right: View of the star's disk from the observation point

D.1 Write down $\mathcal{V}$ as an integral over one variable. It is not necessary to perform the integral explicitly. 1.8 pt

Hint: Integral of some function $f(\theta_y)$, depending only on θ_y, on the surface of the disk of radius θ_R can be represented as:

$$I = \int_{-\theta_R}^{\theta_R} \mathrm{d}\theta_y \int_{-\sqrt{\theta_R^2-\theta_y^2}}^{\sqrt{\theta_R^2-\theta_y^2}} f(\theta_y)\mathrm{d}\theta_x = \int_{-\theta_R}^{\theta_R} 2\sqrt{\theta_R^2-\theta_y^2} f(\theta_y)\mathrm{d}\theta_y.$$

The interferometer is used to observe Betelgeuse. If mirrors **M1** and **M2** are symmetrically separated from the center of **HM** (such that $L_1^* = L_1 + d/2$ and $L_2^* = L_2 + d/2$), then the visibility $\mathcal{V}$ will reset to zero when $d = 0.97\ \mathrm{m}, 3.12\ \mathrm{m}, \ldots$ (the system parameters are the same as in **B.3**). In this part of the problem L_1 and L_2 are not numerically specified, but $\lambda = 500\ \mathrm{nm}$.

D.2 Find the angular radius of Betelgeuse θ_R. Find its radius R, if the distance to it is from the observer is $h = 6.079 \times 10^{15}\ \mathrm{km}$ 0.9 pt

Hint: The first two zeros of the integral

$$F_n = \int_0^1 (1-w^2)^n \cos(\eta w)\,\mathrm{d}w$$

as a function of η are shown in the table below.

n	η_1	η_2
0.5	3.832	7.016
1.0	4.493	7.725
1.5	5.136	8.417

Determination of Refractive Index Gradient and Diffusion Coefficient of Salt Solution from Laser Deflection Measurement (10 points)

Introduction

Diffusion is a process involving random walk of atoms or molecules leading a system towards its thermodynamic equilibrium. For instance, if a vessel contains a mixture of water and a salt solution, there will be a diffusive flux of salt molecules from regions of high salt concentration towards regions of lower salt concentration. The diffusion rate is characterized by the diffusion coefficient . Diffusion plays a major role in a wide range of processes, from biochemistry to astrophysics.

In the following experimental problem, diffusion of salt molecules is studied. Salt molecules will move diffusively from a salt solution towards the region with distilled water, creating a transition layer of variable salt concentration. The refraction index of this solution depends on the salt concentration. Therefore, we can study diffusion process through optical experiments using laser beam deflection method.

Objectives

1. To determine the diffusion coefficient of saltwater solution in water by measuring the gradient of refractive index.
2. To determine the rate of change of the diffusion coefficient to the change of salt solution concentrations.

List of materials

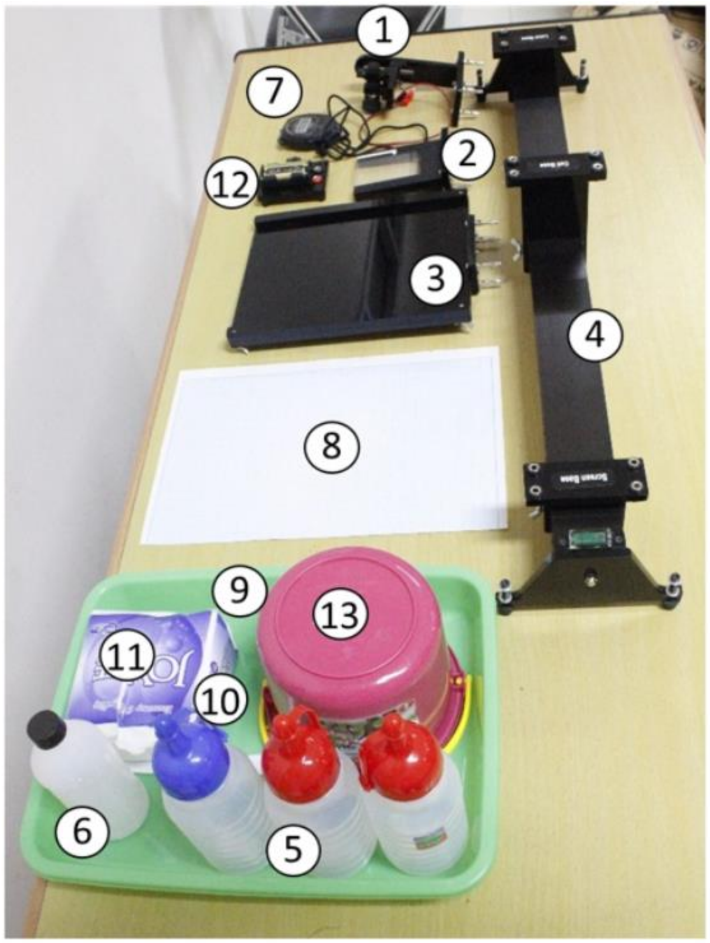

Fig. 1. Materials for this experiment.

1. Line laser module (Diode laser with λ = 632 nm and cylindrical lens)
2. Diffusion cell (6.5 cm x 0.8 cm x 9.5 cm) with holder
3. Screen with holder
4. Optical rail with length scale
5. Saltwater solutions
6. Distilled water (Aquadest)
7. Stopwatch
8. Paper with scale (Block millimeter paper)
9. Pipette (dropper)
10. Knife + Tissue for cleaner
11. Tissue
12. Battery
13. Buckets as waste container of salt and water solution

The schematic diagram of the experimental setup can be seen in **Fig. 2**

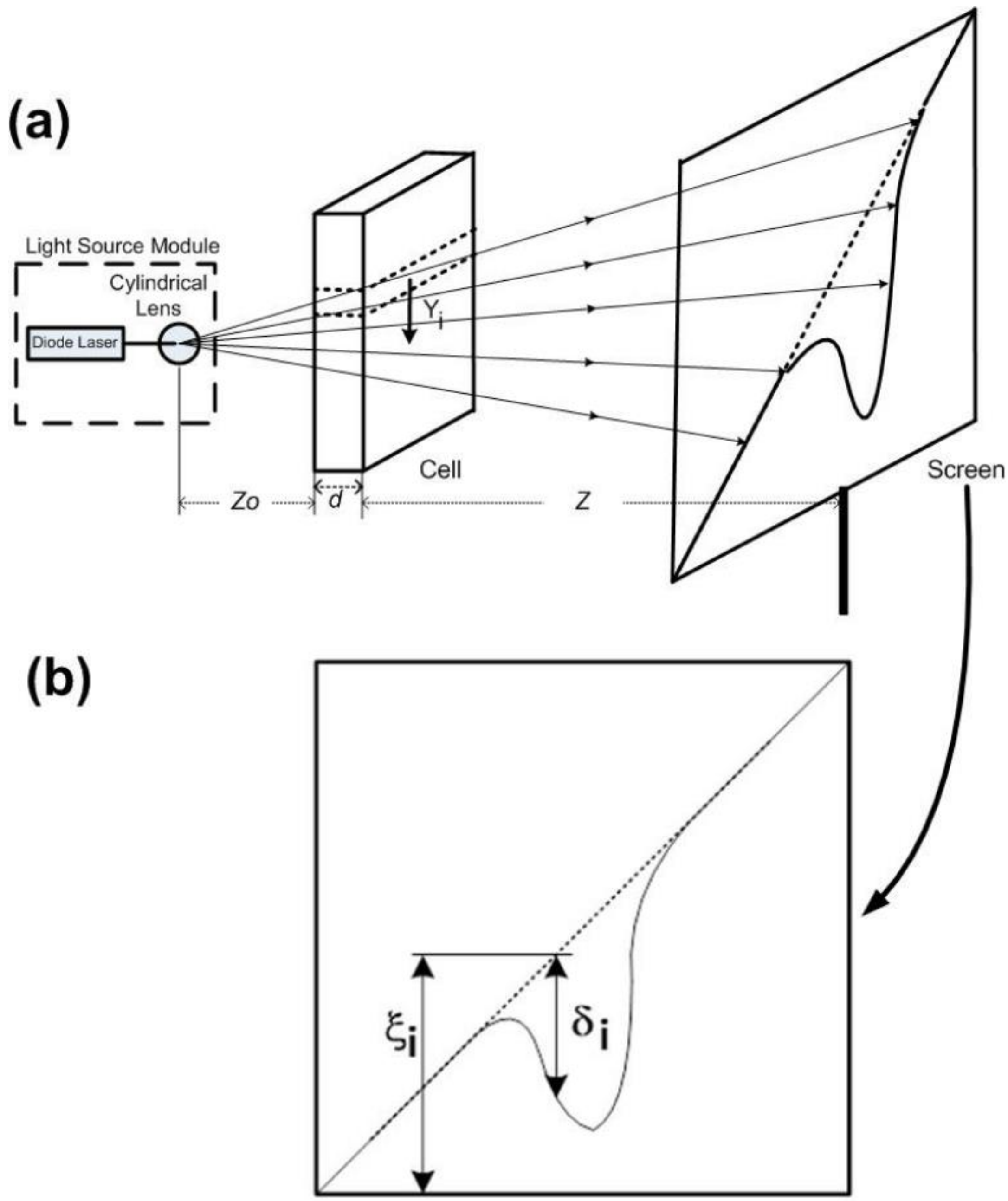

Fig. 2. (a) Schematic diagram of the experiment. **The cell contains saltwater solution with distilled water (aquadest) above it**. **(b)** Typical deflectogram, the deflected laser beam that appears on the screen when the diffusion between the solution 1 and solution 2 occurs.

To obtain a profile of the refractive index gradient as a function of the vertical position in the fluid we must relate the vertical position on the screen (ξ) to the vertical height in the cell (Y), and relate the vertical deflection (δ) to the gradient of refractive index $\left(\frac{dn}{dY}\right)$. From the geometry of the experimental setup (see **Fig. 2**) we have:

$$Y_i = \frac{\xi_i Z_0}{Z_0 + Z + d} \tag{1}$$

where Z_0, Z, and d that are shown in **Fig. 2(a**) denote the distance between the light source module and the diffusion cell, the distance between the diffusion cell and the screen, and the diffusion cell's thickness, respectively. **For measuring Z_0, the line marked on the laser module stand indicates the position of the cylindrical lens.**

The thickness of the cell (d) and the refractive index gradient are both small enough so that the refraction produces negligible vertical displacement of the ray within the cell. In this limit each ray travels at nearly constant vertical height within the cell and is deflected by a single refractive gradient associated with this height. It can be shown that:

$$\left(\frac{dn}{dY}\right)_i = \frac{\delta_i}{Zd} \tag{2}$$

Experimental Procedures:

- In order to obtain the deflected laser trace on the screen (as in **Fig. 2(b)**) you must assemble all the components depicted in **Fig. 1** by following the schematic shown in **Fig. 2(a).**

- Ensure that the laser is on and its spot and its projection on the screen have diagonal form when it hits perpendicularly on the diffusion cell. You may adjust Z, Z_0, and the focal length of the laser (by rotating the back of the laser) in order to obtain a bright and focused line. You can also adjust the direction of the diagonal line on the screw by rotating the laser as a whole (release the laser using the screw on top). In the condition without any water or salt solutions you will see the straight diagonal beam.

- Deflected laser trace will appear when the two different solutions are diffusively mixed. Ensure that you pour the salt solution first into the diffusion cell container up to the limit which is marked by the white line. Drop the water into the container slowly about 40 drops down the side channel using a pipette, after which you may turn on the stop watch to measure the evolution time of diffusion profile. If the setup of Z, Z_0, and the height of the laser are already optimized, the deflected laser trace on the screen will be centered, clear , and the depth of the dip as large as possible. You need to find this optimum setup in order to minimize the error of the measurement.

- After the evolution time of 30 minutes you may draw the laser trace into the millimeter block paper attached to the screen using a pencil. Note that in this experiment, you will be asked to do the measurement for three different salt solution concentrations (i.e. C_0 = 23 g/150 ml , C_0 = 28 g/150 ml, and C_0 = 33 g/150 ml) so that you need to replace the millimeter block frequently. The millimeter block paper should be inserted to the screen and it can be loosened or tightened by rotating the screw attached in the corner of the screen.

- Please make sure that you have written your student code number and the concentration of the solution that you used on this millimeter block paper.

Experiments and Tasks

Part A. Measurement of Refractive Index Gradient of Saltwater Solution (4.5 points)

You need to perform the steps below for all three salt concentrations. Note that no error estimation is needed.

A.1	Perform the experiment to obtain the deflected laser trace on the screen. Duplicate the laser trace into the millimeter block paper attached to the screen using a pencil for the diffusion time (t) of 30 minutes.	1.2 pt

A.2	Measure Z, Z_0, d, ξ_i, and δ_i (with $i = 1, \ldots, 20$ are the data points corresponding to different horizontal positions) from the laser trace in the millimeter block for the diffusion time (t) of 30 minutes. The parameters Z, Z_0, d, ξ_i, and δ_i are stated in cm. Note that Z, Z_0, and d are the same for all measurements. Record the results in the **Table 1**.	1.5 pt

A.3	Calculate Y_i and $\left(\frac{dn}{dY}\right)_i$ (with $i = 1, \ldots, 20$ are the data points) for the diffusion time t of 30 minutes. Plot Y_i vs. $\left(\frac{dn}{dY}\right)_i$ for t = 30 minutes.	1.5 pt

A.4	Determine the Y_i at maximum $\left(\frac{dn}{dY}\right)_i$ obtained from question **A.3**. Assign this Y_i as h.	0.3 pt

Part B. Determination of Diffusion Coefficient (4.2 points)

The curve found in **A.3** can be fitted using the following equations:

$$\left(\frac{dn}{dY}\right)_i = \left(\frac{dn}{dC}\right)\left(\frac{dC}{dY}\right)_i \tag{3}$$

$$\left(\frac{dC}{dY}\right)_i \approx \frac{C_0}{2\sqrt{\pi D t}} \exp\left(-\frac{(h-Y_i)^2}{4Dt}\right) \tag{4}$$

where C, C_0, D, t, and h, denoted as concentration, initial salt concentration, diffusion coefficient, diffusion time, and Y_i at the maximum refractive index gradient $\left(\frac{dn}{dY}\right)$, respectively. Note that $\left(\frac{dn}{dC}\right)$ is constant. The diffusion coefficient can be obtained by using **Eqs. (3)** and **(4)** to form a linear relation between $\left(\frac{dn}{dY}\right)_i$ and Y_i.

B.1	Based on **Eqs. (3)**, **(4),** find such functions $f\left(\frac{dn}{dY}\right)$ and $g(Y)$ that the dependence between $f\left(\frac{dn}{dY}\right)$ and $g(Y)$ would be linear.	0.9 pt

B.2	Make a table (**Table 3** in the answer sheet) that contains the data points abscissa axis, ordinate axis of linear equation from **B.1** for dataset taken from **Part A**. Plot this table.	1.2 pt

B.3	Determine the diffusion coefficient D from the linear plot obtained in **B.2** for data set at t = 30 minutes. Note that linear dependence may hold only for a subrange of your data.	1.5 pt

Part C. Nonlinear Diffusion (1.3 points)

C.1	The analysis above is based on the assumption that D is independent of C. If this is not true, we have the so-called nonlinear diffusion. However, near the maximum of $\frac{dn}{dY}$ we can consider this as an ordinary diffusion, with diffusion coefficient corresponding to the local value of concentration. Determine the rate of change of the diffusion coefficient with the change of salt solution concentration graphically using data from **Part B**.	1.3 pt

Parallel Dipole Line Magnetic Trap for Earthquake and Volcano Sensing (10 points)

Introduction

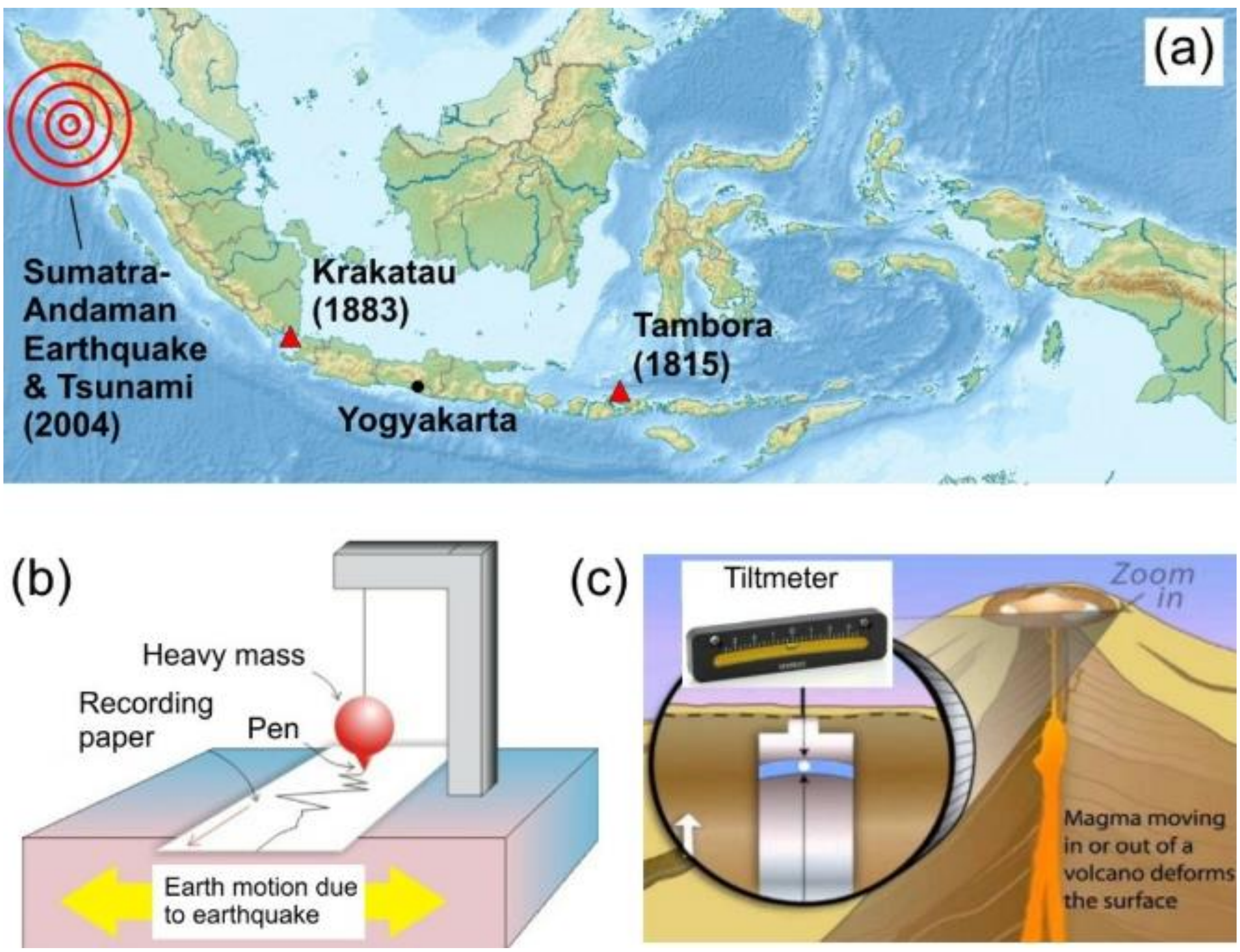

Fig. 1. (a) Map of Indonesia showing its well-known disasters. **(b)** Basic seismometer to detect earthquake. **(c)** Tiltmeter to monitor a volcano.

Indonesia is the world's largest archipelago, with about 17,000 islands sprawling in the tropics and thus, often called the "jewel of the equator". Unfortunately, it has plenty of natural threats such as earthquakes and volcanic eruptions. Colossal catastrophic events (**Fig. 1(a)**) such as the Sumatra-Andaman earthquake and tsunami (2004), Krakatau (1883) and Tambora (1815) volcanic eruptions, are among the deadliest disasters in the recorded history of the world. To detect an earthquake, we use a *seismometer*, usually a pendulum-based system to measure the *ground displacement or acceleration* (**Fig. 1(b)**). To monitor a volcano, we use a *tiltmeter* to detect a change in the *ground inclination* due to underground magma movement (**Fig. 1(c)**). In this problem we will explore the physics and applications of a new kind of magnetic trap and sensor – called the *Parallel Dipole Line* (PDL) trap system – for sensing earthquake and to monitor volcano.

Parallel dipole line system is an arrangement of two linear distribution of magnetic dipole (also called a *dipole line*) as shown in **Fig. 2**. Recently two Indonesian physicists discovered a very interesting effect in this system: If the length of the dipole line is longer than a certain critical length, then the magnetic field becomes stronger on the edges which produces a "*camelback* potential" as shown in **Fig. 2(a)**. ***This** "*camelback* effect" is important as it enables this system to serve as a new type of magnetic trap called *Parallel Dipole Line* (PDL) trap. Experimentally we can realize this PDL trap using a pair of diametric magnets i.e. cylindrical magnets with magnetization along the diameter as shown in **Fig. 2(c)**, where the north and south poles are on the curved sides instead of the flat faces.

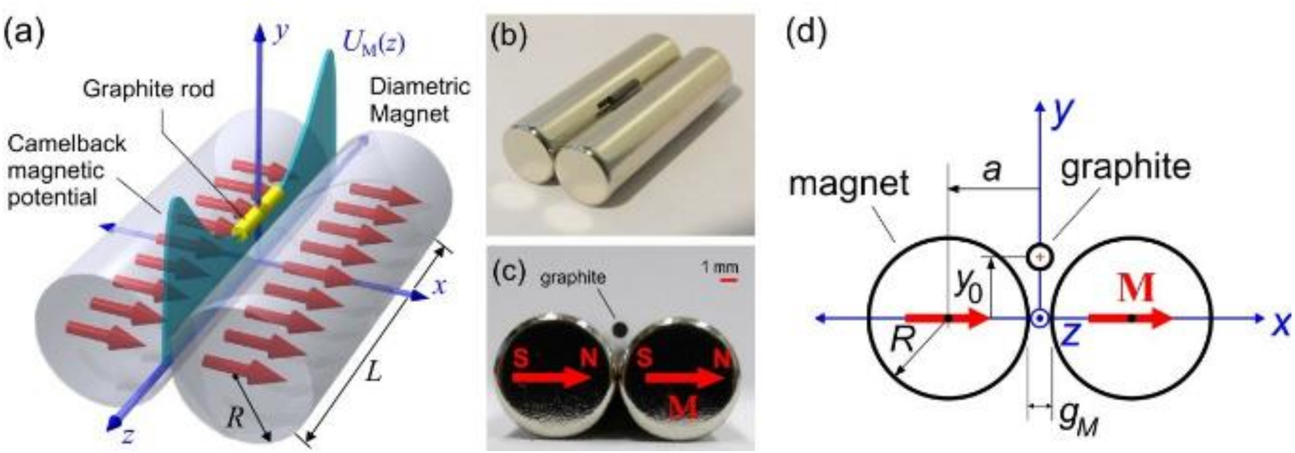

Fig. 2. (a) Parallel dipole line trap model with the camelback potential along the z-direction. **(b)** Experimental setup using "diametric" magnets. **(c)** Cross section view. **(d)** Schematics of the PDL trap. [* Gunawan and Virgus, J. Appl. Phys. 121, 133902 (2017)].

If we drop a graphite rod (ordinary pencil lead) into the trap, it will levitate or get trapped in a stable condition. This occurs because in x-direction the graphite is repelled by the magnets from both sides and in the vertical direction (y), the magnetic repulsion force balances the weight, making it levitate at height (**Fig. 2(d)**). In longitudinal direction (z), the camelback potential holds the graphite stable.

The camelback potential of the magnetic trap serves as a one-dimension oscillator. If you give it a little perturbation along the z-axis to the graphite rod, it will exhibit underdamped oscillation as shown in **Fig.3(a)**. This PDL trap can be used as a sensitive seismometer. If the ground underneath shakes, the graphite rod tends to remain stable and its relative displacement (**Fig. 3(b)**) is the "earthquake" signal. Similarly, it can also be used as a sensitive tiltmeter: If you tilt the trap slightly, the graphite rod will move significantly without any friction.

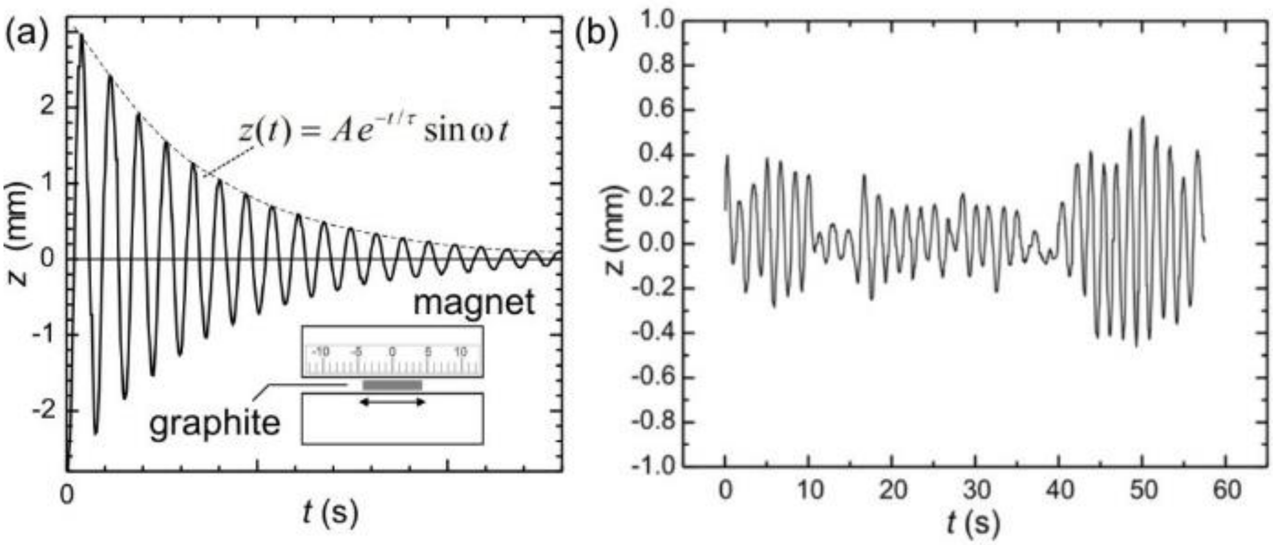

Fig. 3. (a) Underdamped oscillation of a graphite rod along the camelback potential.
(b) Seismometer application: Ground vibration detection by a PDL trap.

Experimental Apparatus

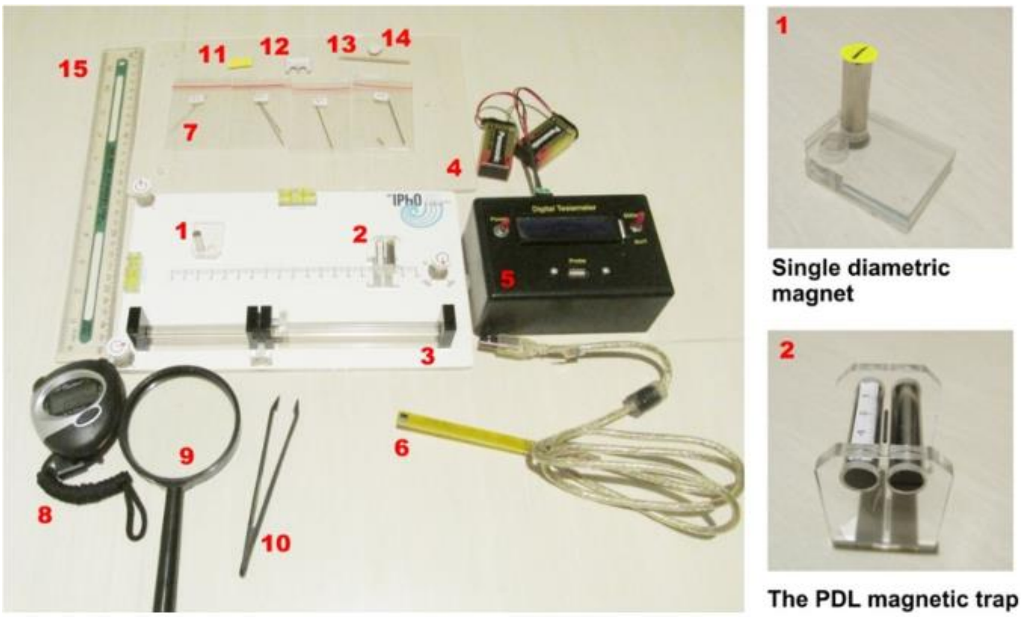

Fig. 4. The experimental setup.

1. Single diametric magnet assembly. Yellow sticker is attached to mark magnetization.
2. The PDL magnetic trap assembly, shown with levitating graphite. Please do not remove the magnets from the assembly.
3. Top platform with three screws.
4. Bottom platform
5. Tesla meter to measure magnetic field. Batteries are provided to power up the Tesla meter, and a cable to connect the Hall probe to the Tesla meter.
6. Hall sensor probe of the Tesla meter.
7. Graphite rods (pencil leads) with four diameter sizes: HB/0.3, HB/0.5, HB/0.7, and HB/0.9. See the constants and data for exact diameters. You may need to break these graphite rods to specific lengths as required.
8. Stopwatch.
9. Magnifying glass.
10. Anti-static tweezers.
11. Round yellow sticker – to mark the magnetization direction (north-south pole) of a single magnet.
12. ”Insert-ruler” to measure graphite levitation height.
13. Toothpick to move the graphite rod around.
14. Silly putty to stick magnet assemblies to platform.
15. Ruler.

Parameters and Constants

Parameter/Constant	Symbol	Value
Radius of the diametric magnet	R	$3.2\ \mathrm{mm}$
Length of the diametric magnet	L	$25.4\ \mathrm{mm}$
Gap of the PDL trap	g_M	$1.5\ \mathrm{mm}$
Mass density of graphite	ρ	$1680\ \mathrm{kg/m^3}$
Graphite rod "HB/0.3" diameter	$d_{0.3}$	$0.38\ \mathrm{mm}$
Graphite rod "HB/0.5" diameter	$d_{0.3}$	$0.56\ \mathrm{mm}$
Graphite rod "HB/0.7" diameter	$d_{0.3}$	$0.70\ \mathrm{mm}$
Graphite rod "HB/0.9" diameter	$d_{0.3}$	$0.90\ \mathrm{mm}$
Room temperature	T	$298\ \mathrm{K}$
Magnetic permeability in vacuum	μ_0	$1.257 \times 10^{-6}\ \mathrm{H/m}$
Boltzmann constant	k_B	$1.381 \times 10^{-23}\ \mathrm{m^2 kg/s^2 K}$
Gravitational acceleration	g	$9.8\ \mathrm{m/s\^{}2}$

NOTE:

1. **Keep the single magnet and the PDL trap (double magnet) assemblies away from each other. They can hit each other and break!**
2. **Turn off the Tesla meter if not in use to save battery!**
3. Detach items 7, 11-14 carefully from the bottom platform (item 4) and then place the top platform (item 3) on the bottom platform.
4. You can use the three screws to adjust the level of the top platform.

Experiment and Questions

Part A. Basic Characteristics of the PDL trap (7.5 points)

[1] Determination of the magnet's magnetization (M) (2.5 points)

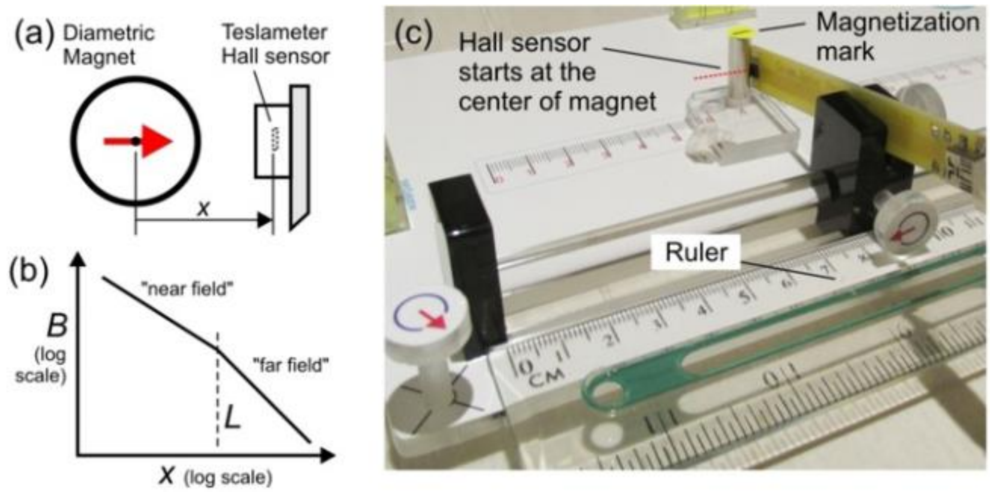

Fig. 5. (a) Measuring the magnetic field. **(b)** Magnetic field profile. **(c)** The setup.

The strength of the magnetic trap depends on the total magnetic dipole moment of the magnet m. It depends on the magnetization M which is the magnetic dipole per unit volume and the characteristics of the magnetic material. For our cylindrical magnet:

$$M = \frac{m}{\pi R^2 L} \tag{1}$$

where R is the radius and L is the length of the magnet (see Parameters and Constants). The value of M is considered the same for all magnets in this experiment. We will study the magnetic field profile and then determine M of the diametric magnet used in our PDL trap.

Take **the single** diametric magnet assembly and setup the experiment as shown in **Fig. 5(c)**. Align the magnetization (as shown in **Fig. 6(a)**) pointing towards the Hall (magnetic field) sensor. Measure the strength of the magnetic field along the x-axis using the Tesla meter. The magnetic field B profile in near field, or the "Dipole line" limit for approximately $x \leq 16$ mm follows:

$$B(x) = \frac{\mu_0 m}{2\pi x^p L} \tag{2}$$

The x-axis is along the magnetization axis of the diametric magnet as shown in **Fig. 6(a)** and x refers to the distance from the center of the diametric magnet to the Hall probe sensor inside the sensor chip. Please address the offset issue shown in **Fig. 6(b).** We will perform measurements only in the "near field" region.

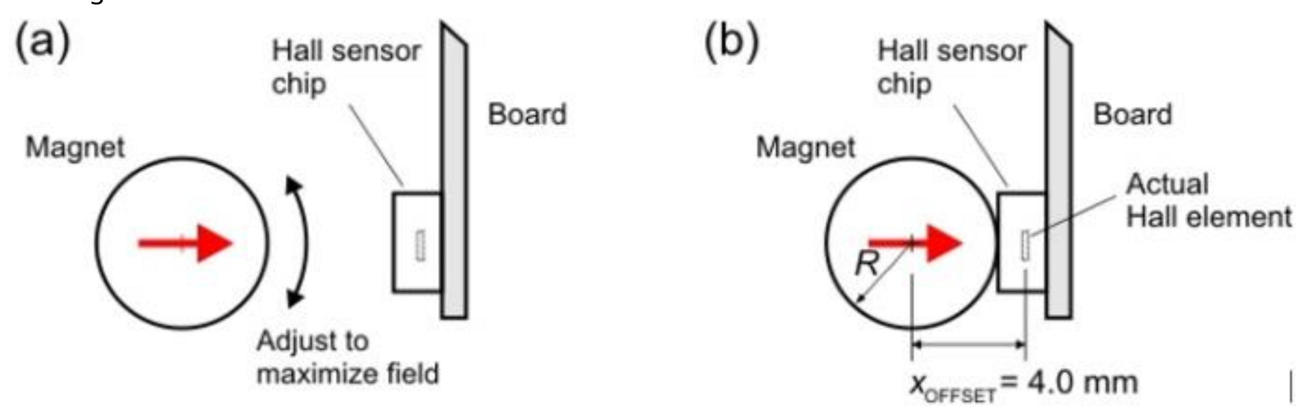

Fig. 6. (a) Adjustments for magnetic field measurement. **(b)** Offset issue to be addressed.

NOTE:

1. **Turn off the Tesla meter if not in use to save battery!**
2. For Tesla meter, wait approximately ~2 sec for each data point before you take readings.
3. Note that x is measured from the center of the magnet. The magnet radius is $R = 3.2$ mm.
4. Use the recommended measurement setup in **Fig. 5(c)**.
5. See **Fig. 6(a)**, adjust the rotation of the magnet so its magnetization points toward the Hall sensor, thus maximizing the field. You can use the yellow round sticker to mark the magnetization direction of the magnet.
6. When the Hall sensor touches the magnet, the actual distance between the center of the magnet and the actual Hall sensor element is the offset value $x_{OFFSET} = 4$ mm.
7. Start your measurement with the Hall sensor at $x = 5$ mm! Don't use the data when the sensor touches the magnet ($x = 4$ mm), as the sensor is saturated or the probe is flexed during the contact.

A.1	Record the offset B_0 of the Tesla meter far away from any magnet. Subtract subsequent field measurements with this value.	0.1 pt

A.2	Measure magnetic field B vs. x in the near field region $(7 \leq x \leq 16\ \mathrm{mm})$! **Where x is the position measured from the center of the magnet.** Record and plot your result on the answer sheet.	1.2 pt

A.3	Use your experimental data to determine the exponent p.	0.7 pt

A.4	Determine the magnet's magnetization M.	0.5 pt

[2] The Magnetic Levitation Effect and Magnetic Susceptibility (χ) (1.0 points)

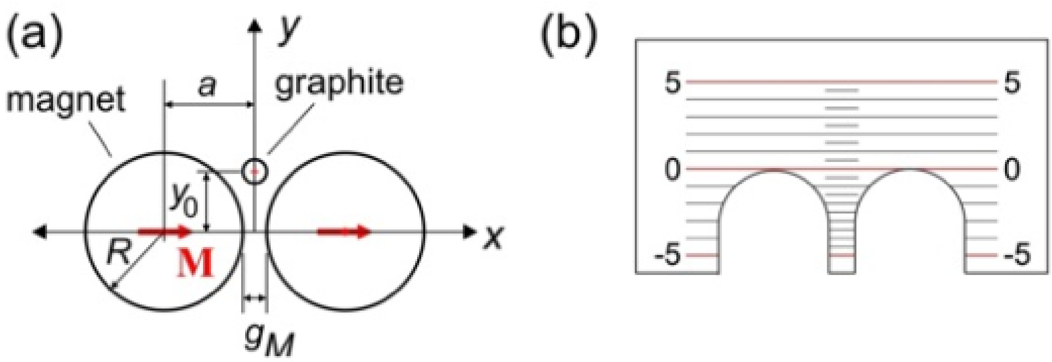

Fig. 7. (a) The magnetic levitation effect in the PDL trap. **(b)** Insert ruler for measuring the levitation height y_0.

The PDL trap demonstrates magnetic levitation effect. The graphite levitates at the center of the trap at height y_0 as shown in **Fig. 7(a)**. The graphite is repelled by the magnet with a force F_M that depends on the magnetic susceptibility χ and the rod position y_0. This magnetic repulsion force on a graphite rod in the PDL trap is given by:

$$F_M(y_0) = -\frac{\mu_0 M^2 \chi V_r}{2} f\left(\frac{y_0}{a}\right) \tag{3}$$

Note that when F_M is positive, the force is directed upwards and there is **a negative sign** in the formula. Here V_r is the volume of the graphite rod, M is the volume magnetization of the magnet (obtained from **A.4**), a is the position of the magnet center given by $a = R + g_M/2$ (see **Fig. 7(a)**) where $g_M = 1.5\ \mathrm{mm}$ is the gap between the magnets. $f(u)$ is the dimensionless function in u for the magnetic repulsion force, and is given by:

$$f(u) = \frac{4u(3-u^2)(1-u^2)}{(1+u^2)^5} \tag{4}$$

A.5	Place the graphite rod HB/0.5 of length $l = 8\ \mathrm{mm}$ gently on the PDL trap. Measure the levitation height y_0 of the rod (see **Fig. 7(a)**) using the insert ruler provided. Press the ruler on the magnets to read the position of the graphite rod.	0.1 pt

A.6	Use the result from **A.5** to determine the magnetic susceptibility χ of the graphite rod.	0.8 pt

A.7	Select the graphite's magnetic material type from the following choices: (i) Ferromagnetic (ii) Paramagnetic (iii) Diamagnetic	0.1 pt

[3] The Camelback Potential Oscillation and Magnetic Susceptibility (χ) (1.0 points)

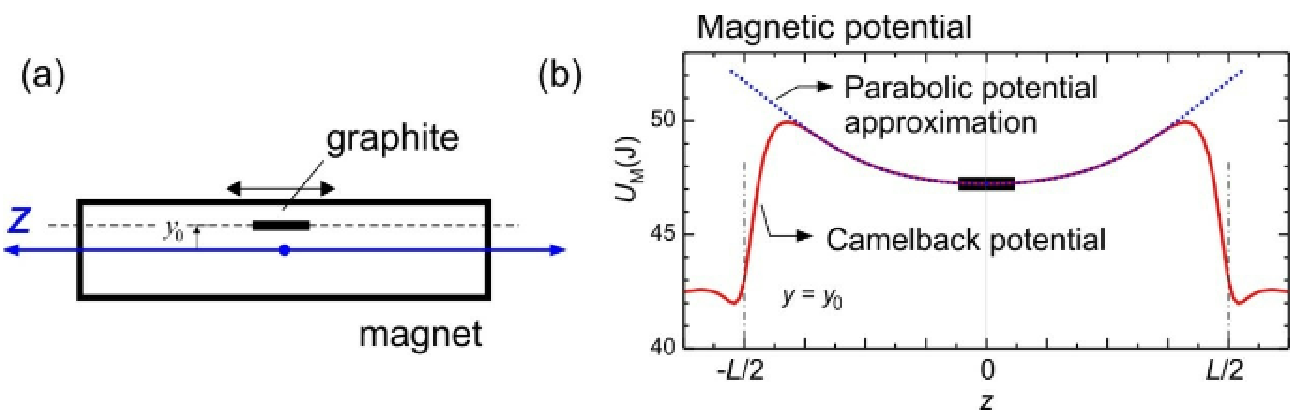

Fig. 8. **(a)** Graphite oscillation at height y_0. **(b)** The camelback potential of the PDL trap and its parabolic approximation.

We will determine χ independently using the oscillation in the magnetic "camelback potential" of the PDL trap as shown in **Fig. 8**. For small amplitudes $(z < 4\ \mathrm{mm})$, the magnetic potential can be approximated as a parabola (shown as the dotted curve in **Fig. 8(b)**):

$$U_M = \frac{1}{2} k_z z^2 \quad (5)$$

where k_z is the spring constant of the potential and z is the center of mass displacement of the rod. This spring constant k_z depends on the magnet magnetization M (from **A.4**) and χ:

$$k_z = -C_1 \mu_0 \chi M^2 V_r \quad (6)$$

where μ_0 is the magnetic permeability, V_r is the volume of the graphite rod, $C_1 = 198.6\ \mathrm{m}^{-2}$ is the constant of this setup.

Drop the graphite rod at the center of the magnetic trap. Adjust the platform level with the screw knobs such that the rod stays at the center of the trap. Displace the rod slightly with a toothpick to induce an oscillation along the camelback potential.

A.8	Perform an oscillation for the "HB/0.5" graphite of length $l = 8\ \mathrm{mm}$. Limit the oscillation amplitude to $A < 4\ \mathrm{mm}$. Determine the oscillation period.	0.2 pt

A.9	Calculate the magnetic susceptibility (χ) of the graphite using the period.	0.8 pt

[4] Oscillator Quality Factor (Q) and Estimation of Air Viscosity (3.0 points)

We observe that the graphite rod oscillation is damped due to the air friction, we want to understand how the friction depends on the dimensions of the graphite rod (diameter and length) and estimate the air viscosity μ_A. The motion of the rod can be modeled as an underdamped oscillation described by $z(t) = Ae^{-t/\tau} \sin \omega t$, as shown in **Fig. 3(a)**, where A is the initial amplitude, ω is the angular frequency and t is the time elapsed.

The amplitude decays with time by a factor of $\exp(-t/\tau)$ where τ is the damping time constant. This determines the oscillator "quality factor" Q defined as $Q = \omega\tau/2$. Oscillation is underdamped if $Q > 0.5$, critically damped if $Q = 0.5$, and overdamped if $Q < 0.5$. This quality factor is important for the design of PDL trap as a seismometer or tiltmeter sensors.

We can calculate the damping time constant τ by approximating the cylinder rods as long ellipsoid and calculate the Stokes drag force. The damping time constant is given by:

$$\tau = \frac{2}{3}\frac{\rho r^2}{\mu_A}\ln\left(\frac{l}{r\sqrt{e}}\right) \tag{7}$$

where ρ, r, and l are the mass density, radius, and length of the graphite rod, μ_A is the viscosity of the air, and e is Euler's number. We will estimate the air viscosity using this model.

A.10	We need to determine the damping time constant τ of oscillation. Sketch your protocol for measuring τ in a simple way.	0.5 pt

A.11	Perform oscillation damping experiments for each rods of various diameters and fixed length of $l = 8\ \mathrm{mm}$. Determine the damping time constant τ for each rods.	1.5 pt

A.12	Determine the air viscosity μ_A.	1.0 pt

Part B. Applications of the PDL trap (2.5 points)

[5] PDL Trap Seismometer (0.5 points)

Imagine you are designing a seismometer using this PDL magnetic trap. For seismometer application we want a very high sensitivity or very low acceleration "noise floor" i.e. the lowest acceleration that it can detect. This acceleration noise floor is given in the unit $\mathrm{m \cdot s^2 \cdot Hz^{1/2}}$ as:

$$a_0 = \sqrt{\frac{4k_B T\omega}{Qm_R}} \tag{5}$$

where k_B is the Boltzmann constant, T is the room temperature (see Parameters and Constants), m_R is the mass of the rod, all in SI units. In **A.11** you have measured τ for several graphite diameters.

B.1	Pick one diameter which serves the best as a seismometer.	0.2 pt

B.2	Calculate the acceleration noise floor a_0 for the rod you've chosen.	0.3 pt

[6] PDL Trap Tiltmeter (2.0 points)

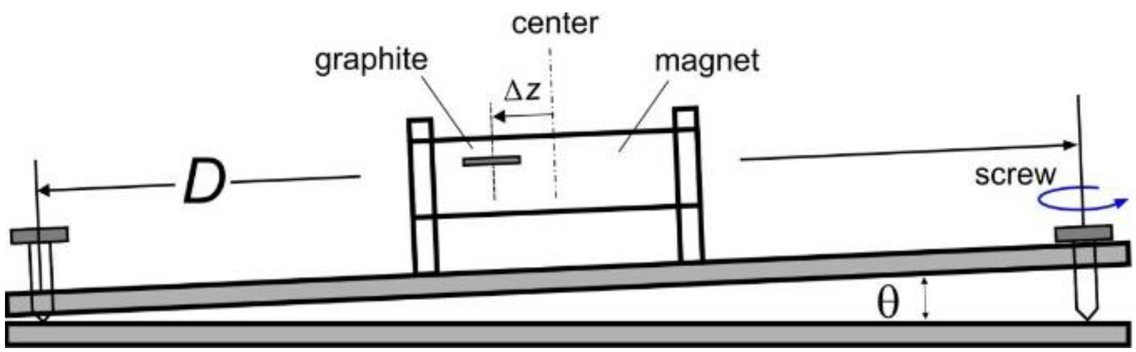

Fig. 9. PDL trap system as a tiltmeter.

We will investigate the PDL trap as a very sensitive tiltmeter to monitor the volcano. The change in the ground inclination is simulated by turning the screw and we want to determine the screw pitch S where S is the change of height per one full turn. We show that by measuring the displacement of the graphite rod in the trap, we can measure the inclination (tilt) precisely

Use pencil rod HB/0.5 of length $l = 8$ mm in this experiment. Start from the center position. Assume the camelback potential can be approximated as a harmonic potential like in **section [3]**.

B.3	Derive the theoretical relation between the displacement Δz with the screw pitch S and the number of turns N.	0.5 pt

B.2	By turning the screw slowly, determine the rod displacement Δz vs. the number of screw turns N. Determine the thread size S.	1.3 pt

B.3	When the ground tilt changes, we want to graphite rod to go to equilibrium as fast as possible to allow for an easy reading. What is the ideal Q factor for a tiltmeter?	0.2 pt