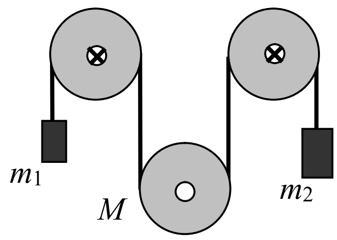
Problem 1: «Three Pulleys».



A light inextensible rope is threaded through three identical pulleys. Two pulleys are fixed (cannot rotate) while the third can frictionlessly rotate about an immobile horizontal axis. The pulleys and the rope lie in the same vertical plane. To prevent the rope from sliding off the pulleys the latter have grooves. Coefficient of friction between a pulley groove and the rope is $\mu = \frac{\ln 2}{\pi}$. The mass of the rotating pulley is $M = 4{,}8$ kg. When evaluating its moment of inertia one can neglect the groove and the central hole and consider the pulley to be a uniform disk. The left end of the rope (see the Figure) is loaded with a weight $m_1 = 200$ g and the right end with a weight $m_2 > m_1$. Initially the system is held at rest.

1) Evaluate the ratio of tensions in the sliding rope on both sides of a fixed pulley. Express your answer via the coefficient of friction and calculate its numerical value. (4 points)
2) Determine the critical value of m_2 required for setting the system in motion after it has been released. Express m_2 in terms of the quantities specified in the problem and write down an explicit formula. (3 points)
3) Evaluate the numerical value of m_2 found above. (1 point).
4) Determine an acceleration of m_2 (absolute value) after the system has been released if its mass is greater by the factor $n = 2$ than the critical mass found in 3). Express the answer in terms of the quantities specified in the problem and write down an explicit formula. (4 points)
5) Express the acceleration found in 4) as a fraction of gravitational acceleration g. (1 point)
6) Determine an acceleration of m_2 (absolute value) after the system has been released if its mass is greater by the factor $k = 4$ than the critical mass found in 3). Express the answer in terms of the quantities specified in the problem and write down an explicit formula. (4 points)
7) Express the acceleration found in 6) as a fraction of gravitational acceleration g. (1 point)
8) Under the condition specified in 6) evaluate a tangential acceleration (absolute value) of the rotating pulley circumference right after the system has been released. Express the answer as a fraction of gravitational acceleration g. (4 points)
9) Suppose a value of m_2 exceeds the critical mass found in 3) and now mass M of the rotating pulley can be varied. Determine the values of M for which a type of relative motion of the rope and the rotating pulley remain the same for any m_2. Write down a formula and express it in terms of the quantities specified in the problem. (4 points)
10) Evaluate the ratio of rope tensions on both sides of the rotating pulley if the value of m_2 is specified in 4) while the pulley mass is less by the factor $k = 4$ than the value specified in the problem. The coefficient of friction between the pulley and the rope remains the same. Write down the answer as a simple fraction. (4 points)

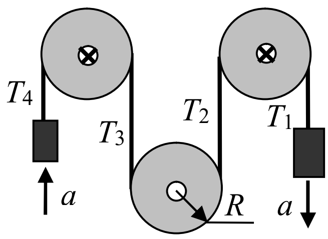
TOTAL score of the problem is 30 points.

TABLE OF ANSWERS:

№	Answer	Maximum score
1	Tension of the «downstream» rope section is greater by the factor of $T_2 / T_1 = e^{\mu\pi} = 2$.	4
2	$m_2 > \overline{m}_2 = e^{2\mu\pi} m_1 = 4m_1$.	3

3	800 g.	1
4	$a = \dfrac{2(n-1)m_1}{2(n+1)m_1 + M \cdot e^{-\mu\pi}} g = \dfrac{4(n-1)m_1}{4(n+1)m_1 + M} g$.	4
5	$\dfrac{g}{9}$.	1
6	$a = \dfrac{ke^{-\mu\pi} - 1}{ke^{-\mu\pi} + 1} g = \dfrac{k-2}{k+2} g$.	4
7	$\dfrac{g}{3}$.	1
8	$a_\tau = \dfrac{2}{9} g$.	4
9	When $M \le 4(e^{2\mu\pi} - e^{\mu\pi})m_1$ or $M \le 8m_1$ the rope does not slide under the rotating pulley for any m_2.	4
10	Tension on the « right» to the rotating pulley is $\dfrac{14}{11}$ of the tension on the « left».	4
Total		**30**

Proposed Solution:



1) If the weights are in motion the rope has to slide over the fixed pulleys. Therefore for any rope element of a length dl in a pulley groove the absolute value of friction force equals $dF_{\text{fr}} = \mu dN$, where dN is a normal force exerted by the pulley. The friction force balances a net tangential force of rope tension dT. The force pressing the rope element to the pulley is the normal component of the net tension force $Td\varphi = dN$. Here $d\varphi = dl / R$ is the angular size of the rope element). Thus $dT = dF_{\text{fr}} = \mu dN = \mu T d\varphi$ and the tension of the rope sliding over a fixed pulley varies according to an exponential law $T(\varphi) = T_0 \cdot e^{\mu\varphi}$.

Suppose that the weight m_2 is moving downward and the rope tensions are labeled in the Figure. Then according to the exponential law $T_2 = \alpha T_1$ and $T_4 = \alpha T_3$ where $\alpha \equiv e^{-\mu\pi} = 0{,}5$.

2, 3) Now let us write down equations of motion of the weights. Since the rope is inextensible the absolute value of a weight acceleration is the same. Let the vertical axis be directed upward. Then one obtains for tensions $T_{2,3}$:

$$\left\{\begin{matrix} m_1 a = T_4 - m_1 g \\ -m_2 a = T_1 - m_2 g \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} T_4 = m_1(g+a) \\ T_1 = m_2(g-a) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} T_3 = \alpha^{-1} m_1 (g+a) \\ T_2 = \alpha\, m_2 (g-a) \end{matrix}\right\}.$$

Concerning the rotating pulley, the rope does not have to slide under it and they can rotate together. If this is the case the net tension dT for any rope element is still balanced by a friction force, although now it is static friction ($dF_{\text{fr}} < \mu dN$). The net torque exerted on the rotating pulley due to friction is $\sum R \cdot dF_{\text{fr}} = R\sum dT = R(T_2 - T_3)$. Besides, if there is no sliding, a tangential acceleration of the pulley circumference equals a linear acceleration a of the weights and rope, so an angular acceleration of the pulley is $\varepsilon = a / R$. Since the pulley moment of inertia can be approximated by $I = \frac{1}{2}MR^2$, the equation of angular motion becomes $\frac{1}{2}MR^2\frac{a}{R} = R(T_2 - T_3)$, whence $T_2 - T_3 = \frac{1}{2}Ma$. Substituting here the tensions found above one obtains the acceleration of weights and rope (providing the rope does not slide under the rotating pulley):

$$\alpha m_2(g-a) - \frac{1}{\alpha}m_1(g+a) = \frac{1}{2}Ma \Rightarrow a = \frac{2(\alpha^2 m_2 - m_1)}{2m_1 + 2\alpha^2 m_2 + \alpha M}g = \frac{m_2 - 4m_1}{4m_1 + m_2 + M}g.$$

This equation is correct if two conditions are met: $a > 0$ (we assume that the system is moving) and $T_3 > \alpha T_2$ (we assume the rope does not slide under the rotating pulley, otherwise $dT = dF_{\text{fr}} = \mu dN = \mu T d\varphi$ and $T_3 = \alpha T_2$). The considered motion is realised if $m_2 > \overline{m}_2 = \alpha^{-2}m_1 = 4m_1 = 800$ g (otherwise friction force holds the system at rest) and if $m_2 < \frac{Mm_1}{\alpha[\alpha^2 M - 4m_1(1-\alpha)]} = \frac{8Mm_1}{M - 8m_1} = 2{,}4$ kg (for a greater m_2 the rope slides under the «middle» pulley). One can see that if m_2 is a bit more than 800 g the above conditions are met and the system starts moving as soon as it has been released.

4, 5) For $m_2 = n\overline{m}_2 = 1{,}6$ kg both inequalities derived above are satisfied, so $a = \frac{2(n-1)m_1}{2(n+1)m_1 + \alpha M}g = \frac{4(n-1)m_1}{4(n+1)m_1 + M}g = \frac{g}{9}$.

6, 7) When m_2 increases to $k\overline{m}_2 = 3{,}2$ kg the second inequality no longer holds. This means that the rope is now sliding under the rotating pulley and $T_3 = \alpha T_2$. The angular acceleration of the pulley is $\varepsilon < a / R$. Using expressions for the tensions derived above one obtains:

$$\alpha^2 m_2(g-a) = \frac{1}{\alpha}m_1(g+a) \Rightarrow a = \frac{\alpha^3 m_2 - m_1}{m_1 + \alpha^3 m_2}g = \frac{k\alpha - 1}{k\alpha + 1}g = \frac{k-2}{k+2}g = \frac{g}{3}.$$

8) Equation of angular motion for the rotating pulley now becomes:

$$\frac{1}{2}MR^2\varepsilon = R(T_2 - T_3) = R(1-\alpha)T_2 \Rightarrow a_\tau = \varepsilon R = \frac{2(1-\alpha)}{M}\alpha m_2(g-a).$$

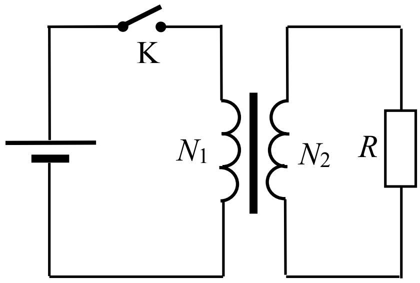
Substituting here the expressions for m_2 and the acceleration one obtains:

$$a_\tau = \frac{4k(1-\alpha)m_1}{\alpha(k\alpha+1)M}g = \frac{8km_1}{(k+2)M}g = \frac{2}{9}g.$$

Obviously this acceleration does not exceed $a = \frac{g}{3}$, as it should be.

9, 10) Notice that if $M \le 4\alpha^{-2}(1-\alpha)m_1 = \alpha^{-2}(e^{2\mu\pi} - e^{\mu\pi})m_1 = 8m_1 = 1{,}6$ kg, the condition $T_3 > \alpha T_2$ holds true for any m_2, i.e. there is no sliding. Therefore acceleration of the «lightweight» pulley can be evaluated according to the equation derived in 4). Substituting $m_2 = 8m_1$ and $M = 6m_1$ one obtains: $a = \frac{2g}{9}$. Then $T_2 = \frac{28}{9}m_1 g$ and $T_3 = \frac{22}{9}m_1 g$, whence $\frac{T_2}{T_3} = \frac{14}{11}$.

Problem 2: «Instantaneous switching-on».

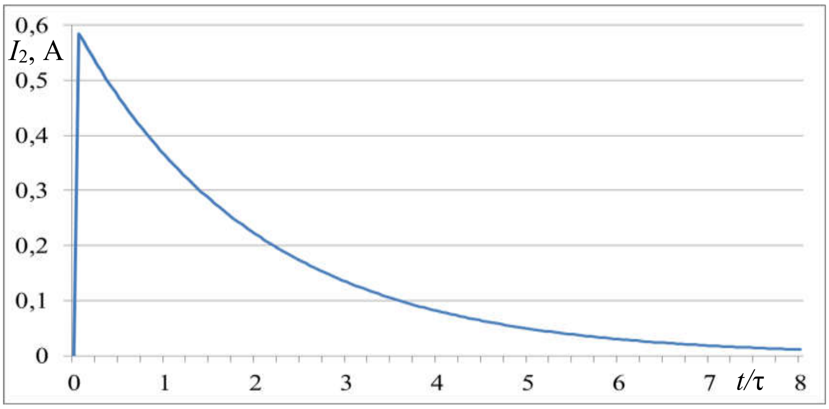


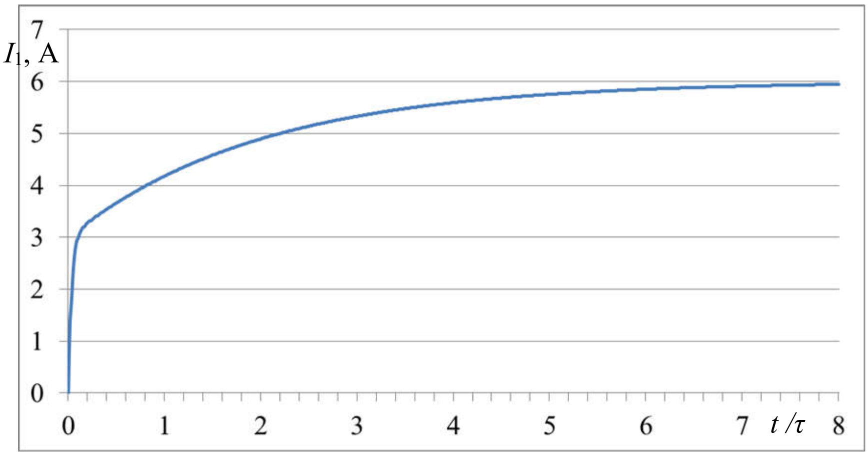
A transformer consists of a toroidal ferromagnetic core (magnetic permeability $\mu >> 1$) of a length $l < 1$ m and cross-sectional area S, a primary winding with the number of turns $N_1 = 20$, and a secondary winding with the number of turns $N_2 = 100$. The transformer is a part of the circuit shown in the Figure. The battery emf equals $\mathcal{E} = 24$ V, its internal resistance is $r = 4$ Ohm. The secondary winding is connected to a resistor of $R = 100$ Ohm. Initially there is no current in the windings and the switch K is open. Then the switch is rapidly closed. Its resistance falls to zero very quickly (although not instantly!) during a time much less than $\tau \equiv \dfrac{\mu_0 \mu N_1^2 S}{lr} = 0{,}05$ sec. Electrical resistance of both windings is much less than 1 Ohm. Dissipation of magnetic flux within the core is negligible.

1) Derive an equation relating a magnetic flux Φ in the core and currents I_1 and I_2 in the primary and secondary windings. Consider a positive direction of current flow in both windings to be the same. (1 point)
2) Consider the circuit including the primary winding and the internal resistance of the battery. Suppose that during switch closing the voltage across the circuit varies as $U_1 = U(t)$. Write down a complete set of first-order differential equations for functions $I_1(t)$ and $I_2(t)$ which describe currents in the primary and secondary windings. (2 points)
3) Specify initial values $I_1(0)$ and $I_2(0)$ (i.e. initial conditions for the set of differential equations derived above). (1 point)
4) Determine a ratio of the current in secondary winding at $t_1 = 2\tau$ to the current at $t_2 = 4\tau$ after the voltage $U(t)$ has been switched on. Evaluate an approximate numerical value of the ratio $\dfrac{I_2(2\tau)}{I_2(4\tau)}$. (3 points)
5) Determine the maximum value of the current in secondary winding after the switch has been closed. Derive an equation for $I_2^{\max}$ in terms of the quantities specified in the problem and evaluate an approximate numerical value of $I_2^{\max}$. (4 points)
6) Determine a time dependence $I_2(t)$ at $t > \tau$ in terms of the quantities specified in the problem. (3 points)
7) Plot approximately the dependence $I_2(t)$ from the moment the switch starts being closed to $T = 0{,}5$ sec. (3 points)
8) What is the maximum current in the primary winding after the switch has been closed? Derive the equation for $I_1^{\max}$ in terms of the quantities specified in the problem and evaluate the numerical value. (2 points)
9) Derive a time dependence $I_1(t)$ for $t > \tau$ in terms of the quantities specified in the problem. (3 points)
10) Plot approximately the dependence $I_1(t)$ from the moment the switch starts being closed to $T = 0{,}5$ sec. (3 points)

TOTAL score for the problem is 25 points.

TABLE OF ANSWERS:

№	Answer	Maximum score
1	$\Phi = \frac{\mu_0 \mu S}{l}[N_1 I_1 + N_2 I_2]$.	1
2	$\begin{cases} U(t) - N_1 \frac{\mu_0 \mu S}{l}\left[N_1 \frac{dI_1}{dt} + N_2 \frac{dI_2}{dt}\right] = I_1 r, \\ -N_2 \frac{\mu_0 \mu S}{l}\left[N_1 \frac{dI_1}{dt} + N_2 \frac{dI_2}{dt}\right] = I_2 R. \end{cases}$	2
3	$I_1(0) = I_2(0) = 0$.	1
4	$\frac{I_2(2\tau)}{I_2(4\tau)} \approx e \approx 2{,}72$.	3
5	$I_2^{\max} \approx \frac{k\varepsilon}{R + k^2 r} \approx 0{,}6$ A. Here $k \equiv \frac{N_2}{N_1} = 5$.	4
6	$I_2(t) \approx -\frac{k\mathcal{E}}{R + k^2 r} \cdot \exp\left[-\frac{R}{(R + k^2 r)\tau} t\right]$.	3
7		3
8	$I_1^{\max} = \frac{\mathcal{E}}{r} = 6$ A.	2
9	$I_1(t) \approx \frac{\mathcal{E}}{r}\left[1 - \frac{R}{R + k^2 r}\exp\left(-\frac{R}{(R + k^2 r)\tau} t\right)\right]$.	3

10		3
Total		**25**

Proposed Solution:

1) Since the core length $l < 1$ m the system can be regarded as quasistationary, so a magnetic flux confined in the core and currents in the windings are related as $\Phi = \frac{\mu_0 \mu S}{l}[N_1 I_1 + N_2 I_2]$ (positive direction in the windings is the same).

2) If the voltage across the primary winding circuit varies during the switch closure as $U_1 = U(t)$, the Kirchhoff's loop rule yields (a winding resistance is negligible):

$$\begin{cases} U(t) - N_1 \dfrac{\mu_0 \mu S}{l}\left[N_1 \dfrac{dI_1}{dt} + N_2 \dfrac{dI_2}{dt}\right] = I_1 r, \\ -N_2 \dfrac{\mu_0 \mu S}{l}\left[N_1 \dfrac{dI_1}{dt} + N_2 \dfrac{dI_2}{dt}\right] = I_2 R. \end{cases}$$

3) Since the current in the windings cannot change instantaneously, the initial condition at $t = 0$ (the moment the switch starts being closed) must read $I_1(0) = I_2(0) = 0$.

4) By multiplying the first equation by N_2 and the second one by N_1 and then subtracting the equations one obtains $I_1 = \frac{U(t)}{r} + \frac{R}{kr} I_2$ at any time (here $k \equiv \frac{N_2}{N_1}$). Substituting this expression into the second equation one obtains the following equation for the current in secondary winding: $\frac{dI_2}{dt} + \frac{R}{R + k^2 r} \cdot \frac{I_2}{\tau} = -\frac{k}{R + k^2 r} \frac{dU}{dt}$. Therefore, as soon as the voltage across the primary winding circuit reaches the battery emf and remains approximately constant thereafter ($\frac{dU}{dt} \approx 0$), the dependence $I_2(t)$ will be approximately given by $I_2(t) \approx C \cdot \exp\left[-\frac{R}{(R + k^2 r)\tau} t\right] = C \cdot \exp\left[-\frac{t}{2\tau}\right]$. The last equation is evaluated by using numerical values of R, r, and k. One can see that $\frac{I_2(2\tau)}{I_2(4\tau)} \approx e \approx 2{,}72$.

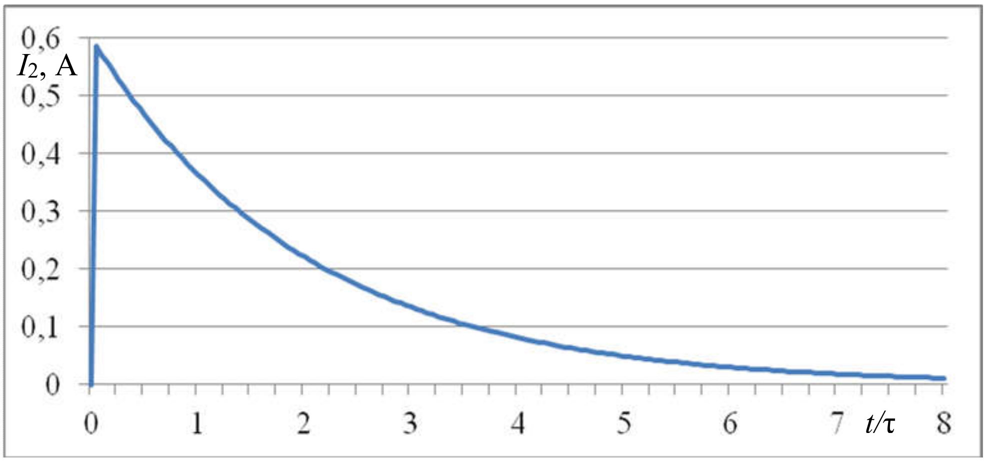
5, 6) However, the above expression does not allow one to determine the final answer for $I_2(t)$ right away since the constant $C \neq 0$ must be specified as the current at the end of the voltage «instantaneous switching-on». To do this one has to specify an appropriate function $U(t)$ such that $U(0) = 0$ and $U(\tau_0) = \mathcal{E}$, where $\tau_0 << \tau$ is a switching-on time of the battery (the time it takes the switch resistance to drop almost to zero). For example consider $U(t) \approx \mathcal{E}\frac{t}{\tau_0}$ during the time interval from zero to τ_0. Then for $t \le \tau_0$ one obtains:

$$\frac{dI_2}{dt} + \frac{R}{R+k^2r}\cdot\frac{I_2}{\tau} \approx -\frac{k}{R+k^2r}\cdot\frac{\mathcal{E}}{\tau_0} \Rightarrow \frac{dI_2}{dt} \approx -\frac{k}{R+k^2r}\cdot\frac{\mathcal{E}}{\tau_0}$$

(the term with I_2 can be neglected since $I_2 \le k\frac{\mathcal{E}}{R}$ and $\frac{\tau_0}{\tau} << 1$), where $I_2(0) = 0$. Thus in this time interval $I_2(t) \approx -\frac{k}{R+k^2r}\cdot\frac{\mathcal{E}}{\tau_0}t$. This implies that at the end of «instantaneous switching-on» the current in secondary winding is opposite to the current in primary winding and it reaches its maximum equal to $I_2^{\max} \approx \frac{k\mathcal{E}}{R+k^2r} \approx 0{,}6$ A. Subsequently the current decays as

$$I_2(t) \approx -\frac{k\mathcal{E}}{R+k^2r}\cdot\exp\left[-\frac{R}{(R+k^2r)\tau}t\right].$$

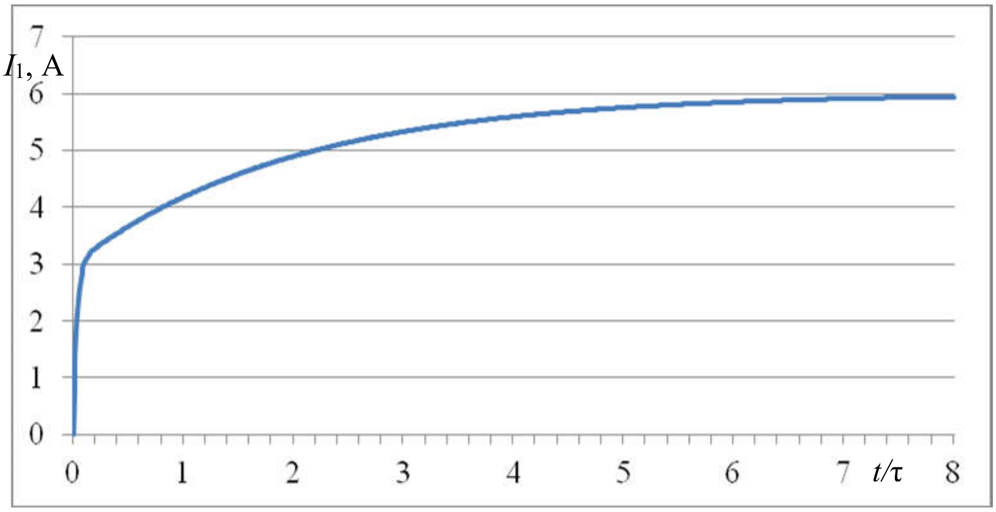
7) Since $\frac{RT}{(R+k^2r)\tau} = 5$ the current in secondary winding at $T = 0{,}5$ sec is less than 0,7% of the maximum value. An approximate plot of $I_2(t)$ is shown in the figure.



8, 9, 10) Using the expression obtained above one finds for the current in primary winding:

$$I_1(t) \approx \begin{cases} \frac{k^2}{R+k^2r}\frac{\mathcal{E}}{\tau_0}t, & t \le \tau_0; \\ \frac{\mathcal{E}}{r}\left[1 - \frac{R}{R+k^2r}\exp\left(-\frac{R}{(R+k^2r)\tau}t\right)\right], & t > \tau_0. \end{cases}$$

Thus after the switching-on starts the current in primary winding rapidly (for $\tau_0 << \tau$) increases to $\frac{k^2\mathcal{E}}{R+k^2r} \approx 3$ A and then exponentially approaches the «stationary» value $\frac{\mathcal{E}}{r} = 6$ A by almost reaching it at the time T. An approximate plot of $I_1(t)$ is shown in the figure.



Notice that the maximum current in primary winding after the switch has been closed (the answer to 8)) can be found simply by analysing the «stationary» mode when the currents are constant and emf in both windings is zero.

Remark: Of course, the linear dependence used to model $U(t)$ of the «instantaneous switching-on» is not a must: any reasonable model gives the same result providing $\tau_0 << \tau$. The linear model is just the simplest. For example one can define $U(t) \approx \mathcal{E}\frac{t}{\tau_0}\left(2-\frac{t}{\tau_0}\right)$ for $t \le \tau_0$ (a smooth crossover to $U(t) \approx \mathcal{E}$ for $t > \tau_0$). Quite effective solution follows if one chooses $U(t) = \mathcal{E} \cdot [1-\exp(-t/\tau_0)]$ (this expression can be used for any t, so the system evolution has not to be split in two stages). In this case a bit more complicated differential equation yields the solution:

$$I_2(t) = \frac{k\mathcal{E}}{R+k^2 r} \cdot \left[\exp\left(-\frac{t}{\tau_0}\right) - \exp\left(-\frac{Rt}{(R+k^2 r)\tau}\right)\right], \qquad (*)$$

$$I_1(t) \approx \frac{\mathcal{E}}{r} \cdot \left[1 - \frac{k^2 r}{R+k^2 r}\exp\left(-\frac{t}{\tau_0}\right) - \frac{R}{R+k^2 r}\exp\left(-\frac{Rt}{(R+k^2 r)\tau}\right)\right]. \qquad (**)$$

One can see that qualitatively time dependence of the currents at $\tau_0 << \tau$ is the same (e.g. the value of $I_2^{\max}$) as in the «linear» model.

Problem 3: «Planet Cronus».

Let us imagine that some civilisation left an artificial planet in the Solar system. The planet orbit lies beyond the orbit of Pluto, so let us call it «Cronus». The Cronus orbit is very close to a circle of radius $R_K = 50$ au (1 astronomical unit (au) is roughly equal to the average distance from the Earth to the Sun), the planet itself is a sphere of radius $r = 5000$ km made of a solid material with a density $\rho \approx 9$ g/cm^3 and good heat conducting properties, and an average heat capacity $c \approx 0{,}3$ J/(g K). A Cronus «year» lasts 20 «solar days», the planet rotates around its axis in the same direction as its orbital rotation around the Sun, and the Cronus rotation axis is perpendicular to its orbital plane. (Notice, that a planet «year» is a period of rotation of its centre of mass around the Sun and «solar day» is the average interval between two «noondays», the moments when the Sun is at maximum elevation on planet sky). Cronus has no satellites, it has an atmosphere consisting of nitrogen, helium, neon, and water vapour. At the beginning of observation Cronus is a rather hot place. This is due to a uniform layer of radioactive material under the planet surface which provides the heat. The material half-life is $\tau_{1/2} = 5000$ Cronus «years». So, one day a human-made probe landed on the Cronus surface and measured temperature and relative humidity of atmospheric layer at the surface. According to the measurements $T_0 \approx 330$ K and $\varphi_0 \approx 80$ %. It turns out the temperature within a dense part of Cronus atmosphere (which contains about 99% of its mass) decreases with altitude as $T(h) \approx T_0 \cdot \left(1 - \frac{h}{6H}\right)$, where $H \approx 10$ km. This distribution remains almost constant (if disturbed the atmosphere returns to this state in about 1 «year»). The atmosphere is so «pure» that all its water remains in vapour state and there are almost no clouds. Necessary constants:

- The gravitational constant $G \approx 6{,}67 \cdot 10^{-11}$ m^3 sec^{-2} kg^{-1}.
- Temperature of the Sun photosphere $T_1 \approx 6000$ K, solar radius $r_1 \approx 0{,}00465$ au, the Stephan-Boltzmann constant $\sigma \approx 5{,}67 \cdot 10^{-8}$ W/(m^2 K^4).
- The specific heat of evaporation of water can be considered to be almost temperature independent and approximately equal to $\kappa \approx 2366$ J/g at T_0. The molar mass of water $\mu \approx 18$ g/mole and the universal gas constant $R \approx 8{,}31$ J/(mole K).
- The boiling point of water under the normal atmospheric pressure of 101,3 kPa equals 373 K.

Using the above information you should be able to answer the following questions (indicate appropriate units of measurement at all numerical results).

1) Determine how many Earth years does a Cronus «year» have. (1 point)
2) Evaluate angular velocity of the planet rotation around its axis relative to the frame of reference of «distant stars». Evaluate a ratio of the planet centripetal acceleration to the free fall acceleration at the planet equator. The answer must consist of two numbers calculated with an accuracy of 1% at least. (2 points)
3) Determine the power of solar radiation absorbed by Cronus by assuming that the planet absorbs all incident radiation. Express the answer in terms of the quantities specified in the problem. (1 point)
4) Evaluate a numerical ratio of the absorbed power to the power radiated by Cronus at the beginning of observations. (2 points)
5) Evaluate a time for which the surface temperature of Cronus would decrease by 4 K if the radioactive «heating» abruptly vanished. Write down the formula in terms of the quantities specified in the problem and numerical value in Cronus years. (3 points)
6) Use the above estimates to develop a mathematical model of the planet surface cooling providing the radioactive heating operates all the time and the surface cools naturally. Determine an approximate time dependence of surface temperature *T*. Write down the formula in terms of the quantities specified in the problem. (3 points)
7) Evaluate a time for which the surface temperature of Cronus would decrease by 4 K (with radioactive heating present). Write down the formula in terms of the quantities specified in the problem. Obtain a numerical value expressed in Cronus years. (2 points)

Further analysis relies on properties of water vapour. Consider two isotherms of water vapour on a pV-diagram at temperatures T and $T + dT$. Let the isotherms correspond to a slow transition from vapour to water and back (it is well known that vapour pressure does not change along such a transition). By «closing» these isotherms near their edges with two small adiabatics you should be able to calculate the following quantities: a) an efficiency of the obtained cycle; b) a work done during the cycle; and c) a heat taken from a «heater» by vapour during the cycle.

8) Using a relation between the above quantities (a-c) and discarding a volume of the liquid phase compared to a volume of vapour of the same mass determine a slope of the temperature

dependence of the saturated water vapour pressure at a given point of T and p_v. The answer is a formula for $\frac{dp_v}{dT}$ as a function of T and p_v. (3 points)

9) Under the same assumptions and considering the heat of evaporation of water to be almost independent of temperature determine a temperature dependence of pressure p_v of saturated water vapour near T_0. Write down the formula. (4 points)

10) Using the obtained results determine the planet surface temperature T_1 at which water vapour begin to condense. Write down the formula expressed in terms of the quantities specified in the problem. Evaluate the numerical value. A loss of water «outside» the atmosphere is negligible. (5 points)

11) Determine the time after beginning of observation when water vapour starts to condense. Write down the formula in terms of the quantities specified in the problem and evaluate the numerical value expressed in Cronus years. (5 points)

12) Evaluate the maximum depth of ocean on the Cronus surface built up of water vapour condensed from the atmosphere. Write down the formula in terms of the quantities specified in the problem, the pressure $p_v(T_0)$ of saturated water vapour at T_0, and a density of liquid water ρ_l. (2 points)

13) Determine (under the same assumptions) a relation between the surface temperature and a ratio of ocean depth to the maximum possible depth at $T < T_1$ but before the complete condensation took place (i.e. when ocean depth is not close to the maximum value). Write down the equation in terms of the quantities specified in the problem. (6 points)

14) Determine the time when the average ocean depth on the Cronus surface becomes a quarter of the maximum. Write down the formula in terms of the quantities specified in the problem and evaluate its numerical value in Cronus years. (4 points)

15) Is it necessary to take into account the heat of vaporisation when evaluating the time of planet cooling at $T < T_1$ if the accuracy of calculation is 5%? And if the accuracy is 0,5%? Answer to both questions «yes» or «no». (2 points)

TOTAL score for the problem is 45 points.

TABLE OF ANSWERS:

№	Answer	Maximum score
1	$\tau_K \approx (353{,}6 \pm 0{,}6)$ Earth years*.	1
2	$\omega \approx (1{,}07 \pm 0{,}02)\cdot 10^{-8}\ \text{c}^{-1}$, $\quad \frac{a}{g} \approx (4{,}56 \pm 0{,}05)\cdot 10^{-11}$.	2
3	$P_{Sun} = \sigma T_1^4 \left(\frac{r_1}{R_K}\right)^2 \pi r^2$.	1
4	$\frac{P_{Sun}}{P_K} = \left(\frac{T_1}{T_0}\right)^4 \left(\frac{r_1}{2R}\right)^2 \approx (2{,}4 \pm 0{,}2)\cdot 10^{-4}$.	2
5	$\Delta\tau_{rad} \approx \frac{\rho c r \lvert \Delta T \rvert}{3\sigma T_0^4} \approx (24 \pm 0{,}5)$ Cronus years.	3

6	$T(t) \approx T_0 \cdot 2^{-t/(4\tau_{1/2})}$.	3
7	$\Delta\tau \approx \frac{4}{\ln 2} \frac{\lvert \Delta T \rvert}{T_0} \tau_{1/2} \approx (350 \pm 3)$ Cronus years.	2
8	$\frac{dp_v}{dT} \approx \frac{\kappa\mu}{RT^2} p_v$.	3
9	$p_v(T) \approx p_v(T_0) \cdot \exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$.	4
10	$T_1 = \dfrac{T_0}{1 - \frac{RT_0}{\kappa\mu} \ln \varphi_0} \approx (325{,}3 \pm 0{,}4)$ K.	5
11	$\tau_1 = \frac{4\tau_{1/2}}{\ln 2} \ln\left[1 - \frac{RT_0}{\kappa\mu} \ln \varphi_0\right] \approx -\frac{4RT_0\tau_{1/2}}{\kappa\mu \ln 2} \ln \varphi_0 \approx (415 \pm 15)$ Cronus years.	5
12	$h_{\max} = \frac{\varphi_0 p_v(T_0)}{\rho_l g}$.	2
13	$\frac{h_l}{h_{\max}} = 1 - \frac{1}{\varphi_0} \exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right] = 1 - \exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_1} - \frac{1}{T}\right)\right]$.	6
14	$\tau_2 = \frac{4\tau_{1/2}}{\ln 2} \ln\left[1 + \frac{RT_0}{\kappa\mu} \ln\left(\frac{4}{3\varphi_0}\right)\right] \approx \frac{4RT_0}{\kappa\mu \ln 2} \ln\left(\frac{4}{3\varphi_0}\right) \tau_{1/2} \approx (950 \pm 25)$ Cronus years.	4
15	No, no.	2
Total		**45**

*An allowed deviation Δ is indicated for any numerical answer: 1 point is awarded if a proposed numerical value falls within the range. If the deviation exceeds Δ and is less than 2Δ, the score is down to 0,5 points. No point is awarded for a numerical answer if the deviation is more than 2Δ.

Proposed Solution:

1) According to the third Kepler's law one obtains that Cronus «year» is related to Earth year as $\frac{\tau_K}{\tau_E} = 50^{3/2} \approx 353{,}6$, i.e. $\tau_K \approx 353{,}6$ Earth years or $\tau_K \approx 1{,}115 \cdot 10^{10}$ sec.

2) The duration of a «solar day» is due both to planet rotation around its axis and to its orbital motion around the Sun. That is, the period of Cronus rotation around its axis with respect to «distant stars» («stellar day») equals 1/19 «year», so $\omega = 19 \frac{2\pi}{\tau_K} \approx 1{,}071 \cdot 10^{-8}$ sec^{-1}. The absolute

value of free fall acceleration near the Cronus surface $g = \frac{GM}{r^2} = \frac{4\pi}{3}G\rho r$ and the absolute value of centripetal acceleration at the equator $a = \omega^2 r$, therefore $\frac{a}{g} = \frac{3\omega^2}{4\pi G\rho} \approx 4{,}56\cdot 10^{-11}$. This means that the planet rotation has almost no influence on dynamical processes near the surface, in particular, one can consider the planet atmosphere to be spherically symmetric.

Remark: the accuracy of numerical values for ω and a/g in the table of answers is chosen to be 2% in order to distinguish the correct quantities from incorrect but numerically close ones which follow if one does not take into account the difference between the period of planet rotation and the solar day, i.e. when using $\omega = 20\frac{2\pi}{\tau_K}$ (the ensuing discrepancy is about 5%).

3, 4) If the radioactive «heater» is switched off, the only heating the planet receives is due to thermal radiation of the Sun, the heating power is $P_{Sun} = \sigma T_1^4 \left(\frac{r_1}{R_K}\right)^2 \pi r^2$; the heat loss is due to thermal radiation by the planet to outer space, its power is $P_K = \sigma T_0^4 \cdot 4\pi r^2$ (the Sun and Cronus are treated as black bodies, so the Stephan-Boltzmann law applies). The ratio of these two quantities is $\frac{P_{Sun}}{P_K} = \left(\frac{T_1}{T_0}\right)^4 \left(\frac{r_1}{2R_K}\right)^2 \approx 2{,}4\cdot 10^{-4}$.

5) Planet temperature decreases by a small amount $\Delta T = -4$ K during a time $\Delta\tau_{rad}$ determined from the equation:

$$C\Delta T \approx \left[\sigma T_1^4 \left(\frac{r_1}{R_K}\right)^2 \pi r^2 - \sigma T_0^4 \cdot 4\pi r^2\right]\Delta\tau_{rad} = -\sigma T_0^4 4\pi r^2 \left[1 - \left(\frac{T_1}{T_0}\right)^4 \left(\frac{r_1}{2R_K}\right)^2\right]\Delta\tau_{rad}.$$

(Here C is the planet heat capacity). The second term in square brackets was found to be $\sim 10^{-4}$ and is negligible compared to 1. This term estimates the contribution of solar radiation to the planet thermal balance. Thus solar radiation has almost no influence on temperature dynamics around T_0 and can be neglected. Notice that this is another argument in favour of the assumption of spherical shape of the atmosphere (thermal radiation of the Sun violates this symmetry). Since $C \approx c\cdot M = \frac{4}{3}\pi\rho c r^3$ one obtains:

$$\Delta\tau_{rad} \approx \frac{\rho c r\,|\Delta T|}{3\sigma T_0^4} \approx 2{,}68\cdot 10^{10}\text{ c} \approx 24 \text{ Cronus years.}$$

This time significantly exceeds the time of relaxation to «quasiequilibrium» altitude profile of the atmosphere and is significantly less that the half-life of radioactive «fuel» of the heating layer.

6, 7) The above results show that the problem of planet cooling can be considered to be spherically symmetric and the cooling process to be «quasiequilibrium» (i.e. it is reasonable to assume that at any time t a heat coming from the planet heating layer to the surface is approximately balanced by the heat $P(t) \approx \sigma T^4(t)$ radiated by the surface to outer space). On the other hand $P(t)$ is directly proportional to the number of radioactive decays per second, so $P(t) = P(0)\cdot 2^{-t/\tau_{1/2}}$. Therefore $T(t) \approx T_0 \cdot 2^{-t/(4\tau_{1/2})}$. Then the temperature drops by $\Delta T = -4$ K for a time determined from the equation $\frac{|\Delta T|}{T_0} \approx 1 - 2^{-\Delta\tau/4\tau_{1/2}} \Rightarrow \Delta\tau \approx \frac{4}{\ln 2}\frac{|\Delta T|}{T_0}\tau_{1/2} \approx 350$ Cronus years. Obviously the inequality $\Delta\tau_{rad} << \Delta\tau$ holds, which validates the assumption that thermal radiation «tunes» the planet temperature in accord with the power of radioactive decay.

Remark: If a contester has not justified spherical symmetry of the problem and quasistationary nature of the process (by comparing $\Delta\tau_{rad}$ and $\Delta\tau$) the maximum score cannot be awarded for items 6 and 7 even if the contester has obtained the correct dependence $T(t)$!

8, 9) The cycle considered is the Carnot cycle with the heater temperature $T + dT$ and the cooler temperature T. The cycle efficiency $\eta = \frac{dT}{T}$. The work done per cycle equals its area, i.e. $A = dp_{\text{v}}(V_v - V_l) \approx dp_{\text{v}} V_v$ (here dp_{v} is the change of pressure of saturated vapour when temperature increases from T to $T + dT$ and a volume of liquid water V_l is negligible compared to a volume of vapour V_v). The heat taken from the heater is the condensation heat of vapour $Q = \kappa m$, so for the cycle efficiency one can write:

$$\frac{dT}{T} = \frac{dp_{\text{v}} V_v}{\kappa m} = \frac{dp_{\text{v}}}{\kappa\rho} \approx \frac{RT}{\kappa\mu p_{\text{v}}} dp_{\text{v}} \Rightarrow \frac{dp_{\text{v}}}{dT} \approx \frac{\kappa\mu}{RT^2} p_{\text{v}}$$

(here the ideal gas law $\rho = \frac{\mu p_{\text{v}}}{RT}$ is used for the vapour). By integrating this equation one obtains: $p_{\text{v}}(T) \approx p_{\text{v}}(T_0) \cdot \exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$.

Remark: Actually this part of the solution is a derivation of the Clausius–Clapeyron relation. If a contester refers to this relation right away, neglects specific volume of water, and uses the ideal gas law to evaluate specific volume of vapour, item 8) should be awarded 2 points out of 3, while item 9) should be awarded the maximum score.

10) It is important to note that since the total mass of water above the planet surface remains constant, the pressure exerted on the planet surface by water does not vary either: $p_0 = \frac{m_{H_2O} g}{4\pi r^2}$. Before condensation it completely due to the pressure of vapour in a surface layer of atmosphere. According to the problem $p_0 = \varphi_0 p_{\text{v}}(T_0)$. The condensation at the Cronus surface begins when this pressure equates the pressure of saturated vapour at a new temperature T_1 such that

$$p_0 = \varphi_0 p_{\text{v}}(T_0) = p_{\text{v}}(T_1) = p_{\text{v}}(T_0) \cdot \exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_0} - \frac{1}{T_1}\right)\right] \Rightarrow \frac{T_0}{T_1} = 1 - \frac{RT_0}{\kappa\mu} \ln\varphi_0,$$

whence $T_1 = \dfrac{T_0}{1 - \frac{RT_0}{\kappa\mu} \ln\varphi_0} \approx (325{,}3 \pm 0{,}4)$ K. Notice that before the onset of condensation the planet cooled by less than 5 K, so the above calculation of the time required to reduce the temperature by $\Delta T = -4$ K is quite realistic.

11) According to the above law of temperature variation, the condensation begins at

$$\tau_1 = \frac{4\tau_{1/2}}{\ln 2} \ln\left[1 - \frac{RT_0}{\kappa\mu} \ln\varphi_0\right] \approx -\frac{4RT_0\tau_{1/2}}{\kappa\mu \ln 2} \ln\varphi_0 \approx 415 \text{ Cronus years.}$$

12) After the condensation begins the pressure $p_0 = \frac{m_{H_2O} g}{4\pi r^2}$ is a sum of the pressure of liquid water layer of a thickness h_l and the pressure of water vapour above its surface which is equal to the pressure of saturated vapour at a current temperature, i.e.

$$p_0 = \varphi_0 p_v(T_0) = \rho_l g h_l + p_v(T_0)\cdot\exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_0}-\frac{1}{T}\right)\right]$$

(ρ_l is the density of liquid water). The maximum ocean depth corresponds is reached when all available water has condensed, i.e. $h_{\max} = \frac{m_{H_2O}}{4\pi r^2 \rho_l} = \frac{\varphi_0 p_v(T_0)}{\rho_l g} \Rightarrow \rho_l g = \frac{\varphi_0 p_v(T_0)}{h_{\max}}$.

13) Therefore ocean depth at a temperature $T < T_1$ is determined from the equation $\varphi_0\left(1-\frac{h_l}{h_{\max}}\right) = \exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_0}-\frac{1}{T}\right)\right]$, whence

$$\frac{h_l}{h_{\max}} = 1-\frac{1}{\varphi_0}\exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_0}-\frac{1}{T}\right)\right] = 1-\exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_1}-\frac{1}{T}\right)\right].$$

14) Ocean depth becomes equal to a quarter of the maximum at $T_2 = \dfrac{T_0}{1+\dfrac{RT_0}{\kappa\mu}\ln\left(\dfrac{4}{3\varphi_0}\right)}$.

Therefore

$$\tau_2 = \frac{4\tau_{1/2}}{\ln 2}\ln\left[1+\frac{RT_0}{\kappa\mu}\ln\left(\frac{4}{3\varphi_0}\right)\right] \approx \frac{4RT_0}{\kappa\mu\ln 2}\ln\left(\frac{4}{3\varphi_0}\right)\tau_{1/2} \approx 949 \text{ Cronus years.}$$

15) Condensation of vapour releases heat. This process, which takes place evenly over the planet surface, is quite «fast», i.e. the heat release «follows» the condensation rate which, in turn, is determined by the rate of planet cooling. Of course, the release of vaporisation heat slows down the planet cooling to some extent and this influence must be analysed. Using the known pressure of saturated water vapour at $T_3 \equiv 373$ K one estimates $p_v(T_0) \approx p_v(T_3)\cdot\exp\left[\frac{\kappa\mu}{R}\left(\frac{1}{T_3}-\frac{1}{T_0}\right)\right] \approx 17$ kPa, so the total vaporisation heat released is $Q_c = \kappa m_{H_2O} = \frac{\kappa\varphi_0 p_v(T_0)\cdot 4\pi r^2}{g} = \frac{3\kappa\varphi_0 p_v(T_0) r}{G\rho} \approx 8\cdot 10^{23}$ J. This heat is released for hundreds of Cronus years on the surface of a planet which material is a good heat conductor. Taking into account the heat capacity of the planet $C \approx \frac{4\pi}{3}\rho c r^3 \approx 1{,}4\cdot 10^{27}$ J/K one can see that its warming due to this process is very small, $\delta T_c \approx 5\cdot 10^{-4}$ K, even if the total amount of the released heat is absorbed. Obviously the planet cooling and the ensuing condensation of vapour is mostly due to reduction of radioactive heating. The temperature decreases approximately according to exponential law. Therefore the rates of cooling and condensation slow down almost at the same pace. Since the condensation lasts no less than cooling before it starts ($\tau_2 > 2\tau_1$), and a temperature change for the considered processes is about $\Delta T \approx 5$ K a decrease in the rate of cooling because of the release of vaporisation heat does not exceed $\frac{\delta T_c}{\Delta T} \approx 10^{-4}$. Therefore the influence of condensation on the cooling is negligible both at an accuracy of 5% and 0,5%.

Remark: Let us recapitulate: the essential part of the analysis of thermal dynamics of Cronus atmosphere is based on understanding of existence of several time scales:

- a scale of weak «fast» effects (like inhomogeneity of solar radiation and release of vaporisation heat) which is about one «day» or less; such effects have almost no influence on the dynamics because they are small and because they get averaged over a time scale of the order of Cronus year and more.

- a scale of «middle rate» relaxation processes (like restoration of the equilibrium temperature profile and the surface temperature) lasting from a Cronus year to decades;
- a scale of «slow» variation of external factors (variation of the heating power) taking hundreds and thousands Cronus years.