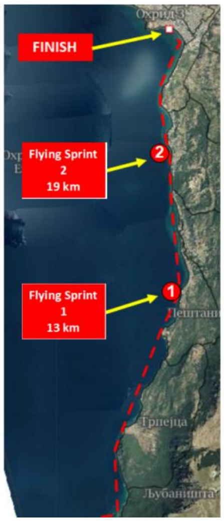


Problem 1. Ohrid Swimming Marathon (25 points). The famous Ohrid Swimming Marathon is an international open water marathon taking place in the waters of Ohrid Lake every year, usually in August. Professional swimmers from all over the world take part in this marathon, swimming the 25 km course. The swimmers start from the place "St. Naum" and finish at the Ohrid port, near the city center. The swimmers' achieved times are measured at three control points, the first is 13 km from the start near the village of Peshtani, the second is 19 km near the hotel "Metropol" and the third one is at the finish line, i.e. 25 km from the start.

Ohrid Marathon 2024 - average swimming speed. The table below shows the officially measured times at the three control points for five randomly selected swimmers in the men's competition at last year's Ohrid Marathon.

a) (**3 points**) Determine the average speed for each swimmer over each section of the route. Express the results in m/s.

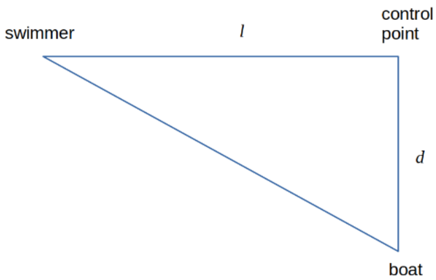
b) (**2 points**) At what average speed (in m/s) did each swimmer swim the entire route?

Swimmer Code	13 km (h:min:s)	19 km (h:min:s)	25 km (h:min:s)
Swimmer 1	3:13:45	4:15:09	5:46:50
Swimmer 2	3:13:05	4:17:37	5:46:52
Swimmer 3	3:13:22	4:15:03	5:50:49
Swimmer 4	3:19:24	4:25:26	6:12:32
Swimmer 5	3:38:58	4:52:20	6:58:27

c) (**1 point**) Is the winner of last year's marathon among the five swimmers in the table, if it is known that he swam the distance to the first checkpoint at an average speed of 1.15 m/s? Explain your answer.

At the Ohrid Swimming Marathon, each swimmer participating in the competition is joined by a team, which follows the swimmer by a boat.

d) (**15 points**) The boats move at a certain distance from the competitors to ensure the smooth running of the competition. At a given moment, one of the swimmers stops and signals (by raising his hand) to his team to give him some water to drink. At that moment, the swimmer is at a distance l from the second control point. After giving the signal, he starts swimming again with an acceleration a_1 towards the control point. At the moment when the swimmer gives the signal, the boat is at a distance d relative to the swimmer's direction of motion, i.e. at a distance d from the control point, as shown in the figure. When the team in the boat sees the signal, the boat starts moving from rest with acceleration a_2. Determine the minimum time t_{min}, for which the boat with the team will reach the swimmer before the swimmer reaches the control point. Assume that the lake water is still.



e) (**2 points**) Express the distances travelled by the boat and the swimmer to the meeting point in terms of the ratio of accelerations, $r = a_2/a_1$, l and d. What is the remaining distance for the swimmer to reach the control point?

f) (**2 points**) Derive a condition under which it is impossible for the boat to meet the swimmer before the control point?

Note: The solution of the quadratic equation of the form $ax^2+bx+c=0$ is given by $x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Solution:

According to the definition, the average speed can be calculated when we divide the distance travelled Δs with the total time Δt needed to pass the distance, i.e.

$$v_{av} = \frac{\Delta s}{\Delta t}$$

a) We will solve this part by using the tabulated data to find the time needed for each swimmer to pass the distance between the control points, 13 km, 6 km и 6 km, correspondingly. Let us first convert the time in second. The results are given in the table bellow:

Swimmer Code	13 km – *t* (s)	19 km – *t* (s)	25 km– *t* (s)
Swimmer 1	11625	15309	20810
Swimmer 2	11585	15457	20812
Swimmer 3	11602	15303	21049
Swimmer 4	11964	15926	22352
Swimmer 5	13138	17540	25107

The times needed to pass the distance between the control points are given in the table below

Swimmer Code	Δt_1(s) - 13 000 m	Δt_2 (s) - 6 000 m	Δt_3(m/s) - 6 000 m
Swimmer 1	11625	3684	5501
Swimmer 2	11585	3872	5355
Swimmer 3	11602	3701	5746
Swimmer 4	11964	3962	6426
Swimmer 5	13138	4402	7567

The average speed calculation is straightforward and we will show the details only for the Swimmer 1.

$$v_{av} = \frac{\Delta s}{\Delta t},\ v_{av}^{1} = \frac{13000\ \text{m}}{11625\ \text{s}} = 1.12\ \text{m/s},\ v_{av}^{2} = \frac{6000\ \text{m}}{3684\ \text{s}} = 1.63\ \text{m/s},\ v_{av}^{3} = \frac{6000\ \text{m}}{3684\ \text{s}} = 1.09\ \text{m/s}.$$

The calculated average speeds for each swimmer along each section are given in Table 1.

Table 1.

Swimmer Code	v_{av} (m/s) - 13 km	v_{av} (m/s) - 6 km	v_{av} (m/s) - 6 km
Swimmer 1	1.1 (1.12)	2 (1.63)	1 (1.09)
Swimmer 2	1.1 (1.12)	2 (1.55)	1 (1.12)
Swimmer 3	1.1 (1.12)	2 (1.62)	1 (1.04)
Swimmer 4	1.1 (1.09)	2 (1.51)	1 (0.93)
Swimmer 5	1.0 (0.99)	1 (1.36)	1 (0.79)

The correct solution of part a) is graded with 3 points. For errors in the numerical calculation of the final value, 0.1 points are subtracted. If the student enters only the final results without showing the details, 1.5 points are awarded.

b) The average speed in the total course can be calculated by dividing the length of the route 25 km by the time duration for each swimmer. For example, for the Swimmer 1, it will be

$$v_{av} = \frac{\Delta s}{\Delta t},\ v_{av} = \frac{25000\ \text{m}}{20810\ \text{s}} = 1.20\ \text{m/s}.$$

The calculated average speeds for each swimmer along the 25 km course are given in Table 2.

Table 2.

Swimmer Code	v_{av}(m/s) - 25 km
Swimmer 1	1.2 (1.20)
Swimmer 2	1.2 (1.20)
Swimmer 3	1.2 (1.19)
Swimmer 4	1.1 (1.12)
Swimmer 5	1.0 (0.996)

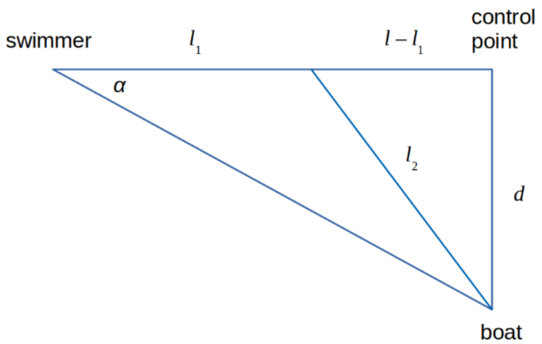
The correct solution of part b) is graded with 2 points. For errors in the numerical calculation of the final value 0.1 points are subtracted. If the student enters only the final results without showing the details, 1.5 points are awarded.

c) If we observe the obtained results for the five swimmers in the first collumn of the Table 1, we will conclude that the value 1.15 m/s (1.2 m/s) is not in the table, which means that the winner is **NOT** among the five swimmers in the provided Table.

The correct answer supported by an explanation is graded by 1.0 point.

Note: *The results given in the tables are rounded to the appropriate number of significant digits. The calculated values rounded to two decimals are given in parentheses.*

d) The boat will reach the swimmer before he/she gets to the control point in a least possible time if it moves along a straight line as it is shown in the following figure:



Correct sketch of the situation (**2.0 points**)

From the law of cosines, we have

$$l_2^2 = l_1^2 + \left(l^2 + d^2\right) - 2l_1\sqrt{l^2+d^2}\cos\alpha \quad \textbf{2.0 points} \quad (1)$$

Moreover,

$$\cos\alpha = \frac{l}{\sqrt{l^2+d^2}} \quad \textbf{2.0 points}$$

and, thus, equation (1) becomes

$$l_2^2 = l_1^2 + \left(l^2+d^2\right) - 2l_1 l \quad \textbf{0.5 point} \quad (2)$$

Since

$$l_1 = \frac{a_1 t^2}{2},\ \ l_2 = \frac{a_2 t^2}{2}, \quad \textbf{2.0 points} \quad (3)$$

for equation (2), we have

$$\frac{a_2^2 t^4}{4} = \frac{a_1^2 t^4}{4} + \left(l^2+d^2\right) - l a_1 t^2,\quad \frac{\left(a_1^2 - a_2^2\right)t^4}{4} + \left(l^2+d^2\right) - l a_1 t^2 = 0. \quad \textbf{1.0 point} \quad (4)$$

By substitution $y = t^2$, this biquadratic equation transforms to the following quadratic equation

$$\frac{\left(a_1^2-a_2^2\right)y^2}{4} - l a_1 y + \left(l^2+d^2\right) = 0. \quad \textbf{1.0 point} \quad (5)$$

which solutions are

$$y_{1,2} = \frac{2la_1 \pm 2\sqrt{l^2 a_1^2 - \left(a_1^2-a_2^2\right)\left(l^2+d^2\right)}}{a_1^2-a_2^2}. \quad \textbf{0.5 points} \quad (6)$$

From here, in order to have a real solution (the discriminant to be positive or equal to zero), it should be satisfied

$$l^2 a_1^2 - \left(a_1^2-a_2^2\right)\left(l^2+d^2\right) \ge 0,\ \text{ i.e., }\ a_1^2 d^2 \le a_2^2\left(l^2+d^2\right). \quad \textbf{2.0 points} \quad (7)$$

Note that the Eq. (6) gives the solution for t^2, so the right hand side should be positive or equal to zero in order to get a real solution for the time. This will be satisfied for the solution with the minus sign in both cases, $a_1 \le a_2$, $a_2 \ge a_1$, So the required solution for the minimum time is y_1 (the one with the minus sign in (6)), i.e.

$$t_{\min} = \sqrt{\frac{2la_1 - 2\sqrt{l^2a_1^2 - \left(a_1^2-a_2^2\right)\left(l^2+d^2\right)}}{a_1^2-a_2^2}}. \quad \textbf{2.0 points} \quad (8)$$

e) Express the distances travelled by the boat and the swimmer to the meeting point in terms of the ratio of accelerations $r = a_2/a_1$, l and d. What is the remaining distance for the swimmer to reach the control point? (**2 points**)

From (3) and (8) we find

$$l_1 = \frac{la_1^2 - a_1\sqrt{l^2a_1^2 - \left(a_1^2 - a_2^2\right)\left(l^2 + d^2\right)}}{a_1^2 - a_2^2} = \frac{l - \sqrt{l^2 - \left(1 - r^2\right)\left(l^2 + d^2\right)}}{1 - r^2},$$ **0.5 points**

$$l_2 = \frac{la_1a_2 - a_2\sqrt{l^2a_1^2 - \left(a_1^2 - a_2^2\right)\left(l^2 + d^2\right)}}{a_1^2 - a_2^2} = \frac{lr - r\sqrt{l^2 - \left(1 - r^2\right)\left(l^2 + d^2\right)}}{1 - r^2},$$ **0.5 points**

$$l - l_1 = \frac{a_1\sqrt{l^2a_1^2 - \left(a_1^2 - a_2^2\right)\left(l^2 + d^2\right)} - la_2^2}{a_1^2 - a_2^2} = \frac{\sqrt{l^2 - \left(1 - r^2\right)\left(l^2 + d^2\right)} - lr^2}{1 - r^2},$$ **1.0 point**

where $r = a_2/a_1$.

f) Derive a condition under which it is impossible for the boat to meet the swimmer before the control point? (**2 points**)

If we analyze the condition (7), under which real solutions for the time exist, we can conclude that if this condition is not satisfied, then the equation (6) will have only imaginary solutions. So the required condition is opposite to condition (7), i.e.

$$a_1^2d^2 > a_2^2\left(l^2 + d^2\right) \Rightarrow \frac{a_2^2}{a_1^2} < \frac{d^2}{l^2 + d^2} = \sin^2\alpha,$$ **1.0 point**

$$r^2 < \sin^2\alpha \Rightarrow r < \sin\alpha \quad \left(0 < \alpha < \pi/2\right).$$ **1.0 point**

Problem 2. *Star wars* (25 points). An artificial satellite with mass $m \ll M$, where M is the mass of the Earth, orbits the Earth on an elliptical trajectory, having the semi-major axis $a = \alpha R$, where R is the mean Earth radius, $\alpha = 3$, and eccentricity $e = 1/2$. Also known is the numerical value of the first cosmic speed, $v_I = 7.91\ \mathrm{km/s}$.

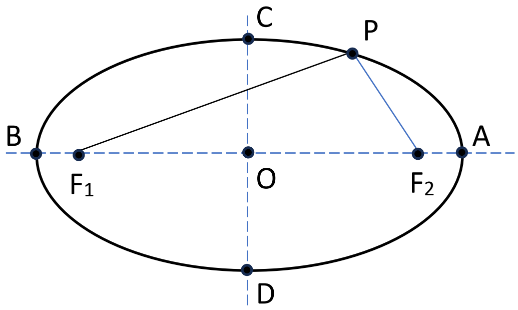
a) (8 points) Determine the speed of the satellite at apogee (point A) (v_a) and at the perigee (point B) (v_p).

A projectile with the mass $m_0 = m/\beta$, where $\beta = 3$, launched from the Earth, moves radially and undergoes a plastic collision with the satellite, when the satellite is at its apogee. After the collision, the eccentricity of the new orbit of the satellite becomes $e' = 3/4$.

b) (2 points) Determine the tangential speed of the satellite immediately after the collision.
c) (7 points) Determine the ratio of the satellite energy immediately after the collision with the projectile to the satellite energy immediately before the collision.
d) (3 points) Determine the radial speed of the satellite immediately after the collision.
e) (5 points) Determine the launching speed of the projectile.

Notes:

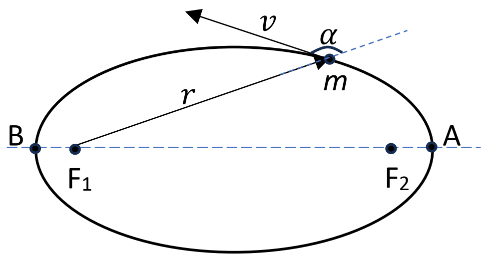
1) *An ellipse is the plane curve represented in the figure, for which* $\mathrm{F_1P} + \mathrm{F_2P} = \mathrm{const.}$*, for any point* P. *The point* O *is the center of the ellipse and the fixed points* $F_{1,2}$ *are the ellipse's focal points. If the Earth is in* F_1*, for example, then* A *is called the apogee and B – the perigee. Other important quantities for an ellipse are:* $a = \mathrm{OA} = \mathrm{OB}$ *– semi-major axis,* $b = \mathrm{OC} = \mathrm{OD}$ *– semi-minor axis,* $c = \mathrm{OF_1} = \mathrm{OF_2}$ *– the linear eccentricity, while the nondimensional parameter characterizing the ellipse is* $e = c/a$*, called the ellipse eccentricity.*



2) *For a satellite – Earth system, there is a physical quantity called angular momentum* L *with respect to the center of the Earth, which is conserved. Its conservation proves the second law of Kepler. Mathematically, the angular momentum has the magnitude (see the drawing below):*

$$L = mvr\sin\alpha,$$

where m *is the satellite mass,* v *– its speed, and* r *– the distance from the center of the Earth (F1) to the satellite.*



Solution:

An artificial satellite with mass $m \ll M$, where M is the mass of the Earth, orbits the Earth on an elliptical trajectory, having the semi-major axis $a = \alpha R$, where R is the mean Earth radius, $\alpha = 3$, and eccentricity $e = 1/2$. Also known is the numerical value of the first cosmic speed, $v_I = 7.91\ \mathrm{km/s}$.

a) (8 points) Determine the speed of the satellite at apogee (v_a) and at the perigee (v_p).

Solution:

The first cosmic speed is the launching speed of a body from the Earth surface to make it orbit the Earth, at its surface:

$$\frac{mv_I^2}{R} = \gamma \frac{mM}{R^2}, \qquad (0.6 \times 2)$$

where γ is the gravitational constant, so

$$v_I = \sqrt{\frac{\gamma M}{R}}. \qquad (0.4)$$

The energy of the satellite on its orbit is

$$E = -\gamma \frac{mM}{2a}. \qquad (0.8)$$

At the apogee

$$E = \frac{mv_a^2}{2} - \gamma \frac{mM}{r_{max}}, \qquad (0.7 \times 2)$$

where

$$r_{max} = a + c = a(1+e), \qquad (0.4 \times 2)$$

so

$$v_a = \sqrt{\frac{\gamma M}{a}\frac{1-e}{1+e}}. \qquad (0.5)$$

or, using the first cosmic speed,

$$\boxed{v_a = v_I\sqrt{\frac{1}{\alpha}\frac{1-e}{1+e}}}. \qquad (0.4)$$

Numerically,

$$\boxed{v_a = \frac{v_I}{3} = 2.636\ \frac{\mathrm{km}}{\mathrm{s}} \cong 2.64\ \frac{\mathrm{km}}{\mathrm{s}}}. (0.3)$$

At perigee, from angular momentum conservation (or from energy equation $\frac{mv_p^2}{2} - \gamma\frac{mM}{r_{min}} = -\gamma\frac{mM}{2a}$)

$$v_p = \frac{r_{max}}{r_{min}} v_a, \qquad (1.0)$$

where

$$r_{min} = a - c = a(1-e). \qquad (0.4)$$

Finally,

$$\boxed{v_p = v_I\sqrt{\frac{1}{\alpha}\frac{1+e}{1-e}}}. \qquad (0.5)$$

Numerically,

$$\boxed{v_p = v_I = 7.91\ \frac{\mathrm{km}}{\mathrm{s}}}. \qquad (0.3)$$

A projectile with the mass $m_0 = m/\beta$, where $\beta = 3$, launched from the Earth, moves radially and undergoes a plastic collision with the satellite, when the satellite is at its apogee. After the collision, the eccentricity of the new orbit of the satellite becomes $e' = 3/4$.

b) **(2 points)** Determine the tangential speed of the satellite immediately after the collision.

Solution:

From momentum conservation on the tangential direction

$$\boxed{v_t' = \frac{mv_a}{m+m_0} = \frac{\beta v_a}{1+\beta}}, \text{(1.4)}$$

Numerically,

$$\boxed{v_t' = \frac{v_I}{4} = 1.9775\ \frac{\text{km}}{\text{s}} \cong 1.98\ \frac{\text{km}}{\text{s}}}. \text{(0.3 x 2)}$$

c) **(7 points)** Determine the ratio of the satellite energy immediately after the collision with the projectile to the satellite energy immediately before the collision.

Solution:

The satellite energy before the collision is

$$E = -\gamma \frac{mM}{2a}.$$

From energy and angular momentum conservation, it follows that

$$e' = \sqrt{1 + \frac{2E'L'^2}{\gamma^2 M^2 (m+m_0)^3}}, \quad \text{(5.0)}$$

where

$$L' = (m+m_0)a(1+e)v_t' = ma(1+e)v_a = m\sqrt{\gamma M a(1-e^2)}. \quad \text{(0.5)}$$

[derivation:

$$E' = \frac{(m+m_0)v^2}{2} - \gamma \frac{(m+m_0)M}{r} \quad \text{(1.0)}$$

and

$$L' = (m+m_0)vr \quad \text{(0.5)}$$

at the apses. Eliminating v between these equations, one gets

$$E' = \frac{L'^2}{2(m+m_0)r^2} - \gamma \frac{(m+m_0)M}{r},$$

or

$$2E'(m+m_0)r^2 + 2\gamma M(m+m_0)^2 \cdot r - L'^2 = 0. \quad \text{(0.5)}$$

The solutions of this quadratic equation are

$$r_{min,max} = \frac{\gamma M(m+m_0)}{2E'}\left[-1 \pm \sqrt{1 + \frac{2E'L'^2}{\gamma^2 M^2 (m+m_0)^3}}\right]. \quad \text{(0.5)}$$

Since

$$r_{min} + r_{max} = 2a' = -\frac{\gamma M(m+m_0)}{E'} \quad \text{(0.5 x 2)}$$

and

$$r_{max} - r_{min} = 2c' = -\frac{\gamma M(m+m_0)}{E'}\sqrt{1 + \frac{2E'L'^2}{\gamma^2 M^2 (m+m_0)^3}}, \quad \text{(0.5 x 2)}$$

then

$$e' = \frac{c'}{a'} = \sqrt{1 + \frac{2E'L'^2}{\gamma^2 M^2 (m+m_0)^3}} \quad \text{(0.5)}$$

end of derivation]

So

$$E' = E\frac{1-e'^2}{1-e^2}\left(1 + \frac{1}{\beta}\right)^3. \quad \text{(0.7)}$$

Under these conditions

$$\boxed{\frac{E'}{E} = \frac{1-e'^2}{1-e^2}\left(1 + \frac{1}{\beta}\right)^3}. \quad \text{(0.5)}$$

Numerically,

$$\boxed{\frac{E'}{E} = \frac{112}{81} = 1{,}38}. \quad \text{(0.3)}$$

d) **(3 points)** Determine the radial speed of the satellite immediately after the collision.

Solution:

Since

$$E' = \frac{(m+m_0)v'^2}{2} - \gamma\frac{(m+m_0)M}{a(1+e)}, \qquad (0.5)$$

then

$$v'^2 = \frac{v_I^2}{\alpha}\left[\frac{2}{1+e} - \frac{1-e'^2}{1-e^2}\left(1+\frac{1}{\beta}\right)^2\right]. \qquad (1.0)$$

Under these conditions,

$$v_r'^2 = v'^2 - v_t'^2, \qquad (0.7)$$

or

$$v_r'^2 = \frac{v_I^2}{\alpha(1-e^2)\beta^2(1+\beta)^2}\left[e'^2(1+\beta)^4 - (1+2\beta+e\beta^2)^2\right],$$

which gives

$$v_r' = \frac{v_I}{\beta(1+\beta)}\sqrt{\frac{e'^2(1+\beta)^4-(1+2\beta+e\beta^2)}{\alpha(1-e^2)}}. \qquad (0.5)$$

Numerically,

$$v_r' = 5.058\ \frac{\mathrm{km}}{\mathrm{s}} \cong 5.06\ \frac{\mathrm{km}}{\mathrm{s}}. \qquad (0.3)$$

e) **(5 points)** Determine the launching speed of the projectile.

Solution:

From momentum conservation for the collision, on the radial direction, it follows that

$$v_p = v_r'\frac{m+m_0}{m_0} = v_r'(1+\beta), \qquad (1.0 + 0.5)$$

or

$$v_p = \frac{v_I}{\beta}\sqrt{\frac{e'^2(1+\beta)^4-(1+2\beta+e\beta^2)}{\alpha(1-e^2)}}. \qquad (0.5)$$

The energy conservation for the projectile before the collision is written

$$\frac{m_0v_{p0}^2}{2} - \frac{\gamma m_0M}{R} = \frac{m_0v_p^2}{2} - \frac{\gamma m_0M}{a(1+e)}, \qquad (1.5)$$

so

$$v_{p0} = \sqrt{v_p^2 + 2v_I^2\left(1-\frac{1}{\alpha(1+e)}\right)}. \qquad (0.7)$$

Finally,

$$v_{p0} = v_I\sqrt{\frac{e'^2(1+\beta)^4-(1+2\beta+e\beta^2)}{\alpha(1-e^2)\beta^2} + 2\left(1-\frac{1}{\alpha(1+e)}\right)}. \qquad (0.5)$$

Numerically,

$$\boxed{v_{p0} = 22.5\ \frac{\mathrm{km}}{\mathrm{s}}}. \qquad (0.3)$$

Problem 3. Charged spheres and springs (25 points). Two identical small spheres, each of mass $m = 1.00$ g, are electrically charged with the same unknown charge q, and are suspended from the same point by two identical, inextensible wires of length $L_0 = 0.200$ m. Each wire forms an angle of $\alpha = 30°$ with the vertical. Wires are non-conducting.

a) **(2 points)** Find the mathematical expression of the electrostatic force acting on each sphere, as well as its numerical value.

b) **(2.5 points)** Find the mathematical expression of the electric charge on each sphere, in terms of the known quantities, as well as its numerical value.

c) **(3.5 points)** Find the mathematical expression of the energy stored in the system, assuming the gravitational potential energy is zero at a distance L_0 below the point from which the wires are suspended.

Now consider 5 identical spheres, each with the same properties and electric charge as above. Four of them are fixed on a horizontal surface, at the corners of a square of side L_0. The fifth sphere is attached to the other four by four identical springs, of negligible mass, natural (unstretched) length L_0, and unknown spring constant k. The springs remain straight (they do not bend). The system forms a regular square pyramid. When the system is in equilibrium, each spring is compressed by a distance ΔL. The springs are non-conducting.

d) **(5 points)** Find the mathematical expression of the spring constant k, in terms of $m, q, L_0, \Delta L$, and known constants.

e) **(2 points)** Find the mathematical expression of the electrostatic energy of the system in terms of $q, L_0, \Delta L$ and known constants.

f) **(10 points)** The sphere at the top of the pyramid is displaced vertically by a very small distance h, where $h \ll L_0$. Show that the restoring force acting on the sphere immediately after it is released is of the form $F_r = C \cdot h$ where C is a constant. Find the mathematical expression for the constant C.

Note: The electric permittivity of the medium $\varepsilon_0 = 8.854 \cdot 10^{-12}$ F/m and the gravitational acceleration $g = 9.80$ ms^{-2} are known.

Solution:

a)

$\frac{F_e}{mg} = \tan\alpha$ (1.5)

$F_e = mg \cdot \tan\alpha$ (0.2)

$F_e = 5{,}66 \cdot 10^{-3}$ N. (0.3)

b)

$F_e = \frac{q^2}{4\pi\varepsilon_0 d^2}$ (1.0)

$d^2 = 4L_0^2 sin^2\alpha$ (0.7)

$q = \pm 4L_0 sin\alpha \cdot \sqrt{\pi\varepsilon_0 F_e}$ (0.5)

$q = \pm 159$ nC (0.3)

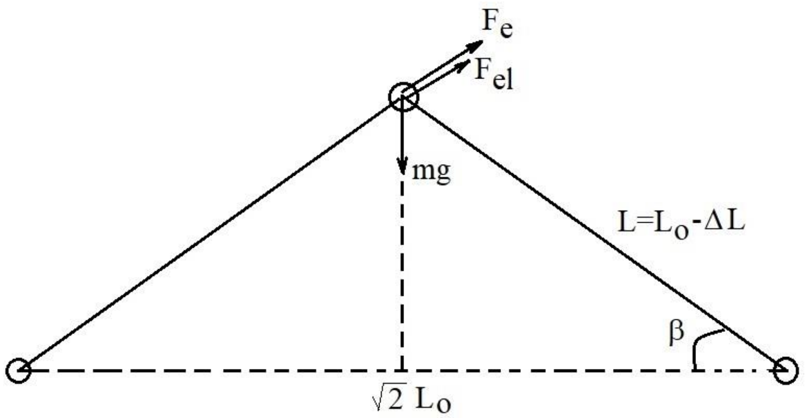
c)

$W_{pg} = 2mgL_0(1 - cos\alpha)$ (1.5)

$W_e = \frac{q^2}{8\pi\varepsilon_0 L_0 sin\alpha}$ (1.5)

$W = W_{pg} + W_e = 2mgL_0(1 - cos\alpha) + \frac{q^2}{8\pi\varepsilon_0 L_0 sin\alpha}$ (0.5)

d)



(1.5 for the drawing)

$F_e = \frac{q^2}{4\pi\varepsilon_0 L^2}$ (0.5)

$F_{el} = k \cdot \Delta L$ (0.5)

$mg = 4\left(\frac{q^2}{4\pi\varepsilon_0 L^2} + k \cdot \Delta L\right) sin\beta$ (1.0)

$mg = 4\left(\frac{q^2}{4\pi\varepsilon_0 (L_0-\Delta L)^2} + k \cdot \Delta L\right) sin\beta$ (0.5)

$$mg = 4\left(\frac{q^2}{4\pi\varepsilon_0 (L_0-\Delta L)^2} + k \cdot \Delta L\right)\sqrt{1-\frac{L_0^2}{2L^2}}$$

$$k \cdot \Delta L = \frac{mg}{4} \cdot \left(1-\frac{L_0^2}{2L^2}\right)^{-\frac{1}{2}} - \frac{q^2}{4\pi\varepsilon_0 (L_0-\Delta L)^2}$$

$k = \frac{mg}{4 \cdot \Delta L} \cdot \left(1-\frac{L_0^2}{2(L_0-\Delta L)^2}\right)^{-\frac{1}{2}} - \frac{q^2}{4\pi\varepsilon_0 \cdot \Delta L (L_0-\Delta L)^2}$ (1.0)

e)

$W_e = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{4}{L_0-\Delta L} + \frac{4}{L_0} + \frac{2}{\sqrt{2}L_0}\right]$ (1.5)

$W_e = \frac{q^2}{2\pi\varepsilon_0}\left[\frac{2}{L_0-\Delta L} + \frac{2}{L_0} + \frac{1}{\sqrt{2}L_0}\right]$ (0.5)

f)

If the ball at the vertex descends on a very small distance h, the length of the springs decreases with $hsin\beta$. (0.5)

The supplementary vertical component of the elastic is:

$$F_1 = 4ksin^2\beta \cdot h \qquad (1.0)$$

The new electric force will be:

$$F_e' = \frac{q^2}{4\pi\varepsilon_0(L-hsin\beta)^2} \qquad (1.0)$$

But

$$\frac{1}{L-hsin\beta} = \frac{L+hsin\beta}{L^2-(hsin\beta)^2} \cong \frac{L+hsin\beta}{L^2} \qquad (1.0)$$

$$\frac{1}{(L-hsin\beta)^2} \cong \frac{(L+hsin\beta)^2}{L^4} = \frac{L^2+2Lhsin\beta+(hsin\beta)^2}{L^4} \cong \frac{L^2+2Lhsin\beta}{L^4} \qquad (1.0)$$

so:

$$F_e' \cong \frac{q^2}{4\pi\varepsilon_0} \cdot \frac{L^2+2Lhsin\beta}{L^4} \qquad (0.5)$$

The supplementary vertical component of the electric force is:

$$F_2 = 4(F_e' - F_e)sin\beta \qquad (1.0)$$

$$F_2 \cong 4\left(\frac{q^2}{4\pi\varepsilon_0} \cdot \frac{L^2+2Lhsin\beta}{L^4} - \frac{q^2}{4\pi\varepsilon_0 L^2}\right)sin\beta$$

$$F_2 \cong 4\frac{q^2}{4\pi\varepsilon_0 L^2}\left(\frac{L^2+2Lhsin\beta}{L^2} - 1\right)sin\beta$$

$$F_2 \cong 4\frac{q^2}{4\pi\varepsilon_0 L^2} \cdot \frac{2hsin^2\beta}{L}$$

$$F_2 \cong \frac{2q^2}{\pi\varepsilon_0} \cdot \frac{sin^2\beta}{L^3} \cdot h \qquad (1.5)$$

The restoring force is:

$$F_r = F_1 + F_2 \cong 4ksin^2\beta \cdot h + 4\frac{2q^2}{4\pi\varepsilon_0} \cdot \frac{sin^2\beta}{L^3} \cdot h \qquad (0.5)$$

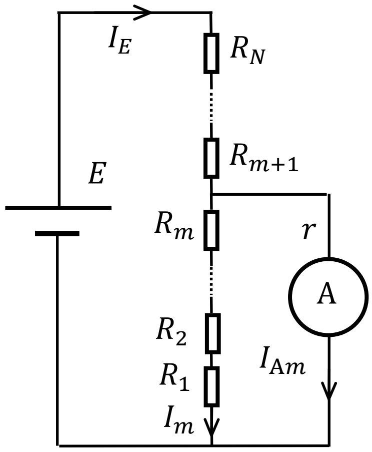
$$F_r \cong 4sin^2\beta\left(k + \frac{q^2}{2\pi\varepsilon_0 L^3}\right) \cdot h \qquad (1.0)$$

$$F_r = C \cdot h, \text{ with } C \cong 4sin^2\beta\left(k + \frac{q^2}{2\pi\varepsilon_0 L^3}\right)$$

$$sin^2\beta = \frac{L^2 - \frac{L_0^2}{2}}{L^2} = 1 - \frac{L_0^2}{2(L_0-\Delta L)^2} \qquad (0.5)$$

$$C \cong 4\left(1 - \frac{L_0^2}{2(L_0-\Delta L)^2}\right)\left(k + \frac{q^2}{2\pi\varepsilon_0 L^3}\right) \qquad (0.5)$$

Problem 4. *N* resistors and an ammeter (25 points). A circuit diagram is given in the figure. It contains a battery with voltage E, N resistors of equal value $R_1 = R_2 = \cdots = R_N = R$, and an ammeter with resistance r.



a) (5 points) Obtain a formula for the current I_{Am} through the ammeter expressed in terms of the given quantities E, R, r, N, m.

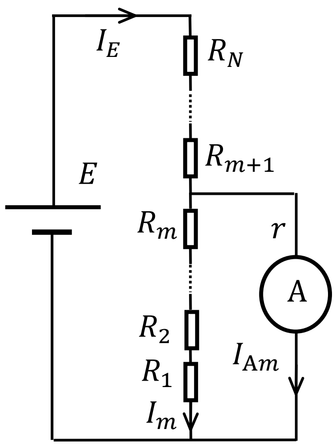
b) (13 points) In a circuit with $N = 50$ and E =12 V a series of measurements of the ammeter current I_{Am} were made at 10 consecutive values of m $(m, m+1, m+2, \ldots, m+9)$. These values are given in the table below:

k	I_{Am}, mA		
m	8.63		
$m+1$	9.16		
$m+2$	9.77		
$m+3$	10.5		
$m+4$	11.2		
$m+5$	12.2		
$m+6$	13.3		
$m+7$	14.5		
$m+8$	16.1		
$m+9$	18.1		

Using suitable variables represent the measurements on such a graph that the values of resistance R and number m can be obtained. Assume that $m \gg 1$ and $r < R$. Draw the dependence on the graph paper provided with the answer sheets. All additionally calculated data must be present in a table on the answer sheets provided. Use as many columns in the table in your answer sheets as you need. Calculate R and m. Round the value of R to a multiple of 5.

c) (7 points) Calculate r. Any additional calculated data should be presented in the table. Round the value of r to a multiple of 5.

Solution



From the circuit it is seen that $rI_{\mathrm{Am}} = mRI_m$ **(1.0 p)** (1)

and $I_E = I_{\mathrm{Am}} + I_m$. **(0.5 p)** (2)

Therefore $I_m = I_{\mathrm{Am}} \frac{r}{mR}$ (3)

and $I_E = I_{\mathrm{Am}} \left(1 + \frac{r}{mR}\right)$. **(1.0 p)** (4)

Also $E = (N - m)RI_E + rI_{\mathrm{Am}}$. **(1.0 p)** (5)

Substituting (4) in (5),

$$E = (N - m)RI_{\mathrm{Am}} \left(1 + \frac{r}{mR}\right) + rI_{\mathrm{Am}} = I_{\mathrm{Am}} \left[(N - m)R \left(1 + \frac{r}{mR}\right) + r\right]. \quad (6)$$

Finally, $I_{\mathrm{Am}} = \frac{E}{(N-m)R\left(1+\frac{r}{mR}\right)+r}$. **(1.5 p)** (7)

b) (13 points) *In a circuit with $N = 50$ a series of measurements of the ammeter current I_{Am} were made at 10 consecutive values of m ($m, m+1, m+2, \dots, m+9$). These values are given in the table below:*

k	I_{Am}, mA	$\frac{E}{I_{\text{Am}}}$, Ω	$k = N - \frac{E}{I_{Am}R}$	$r = \frac{m}{N}\left[\frac{E}{I_{\text{Am}}} - (N-m)R\right]$, Ω
m	8.63	1391	32.6	20.1
$m+1$	9.16	1310	33.6	20.5
$m+2$	9.77	1228	34.6	19.8
$m+3$	10.5	1143	35.7	16.5
$m+4$	11.2	1071	36.6	23.3
$m+5$	12.2	983.6	37.7	17.9
$m+6$	13.3	902.3	38.7	17.4
$m+7$	14.5	827.6	39.7	22.1
$m+8$	16.1	745.3	40.7	20.8
$m+9$	18.1	663.0	41.7	19.3

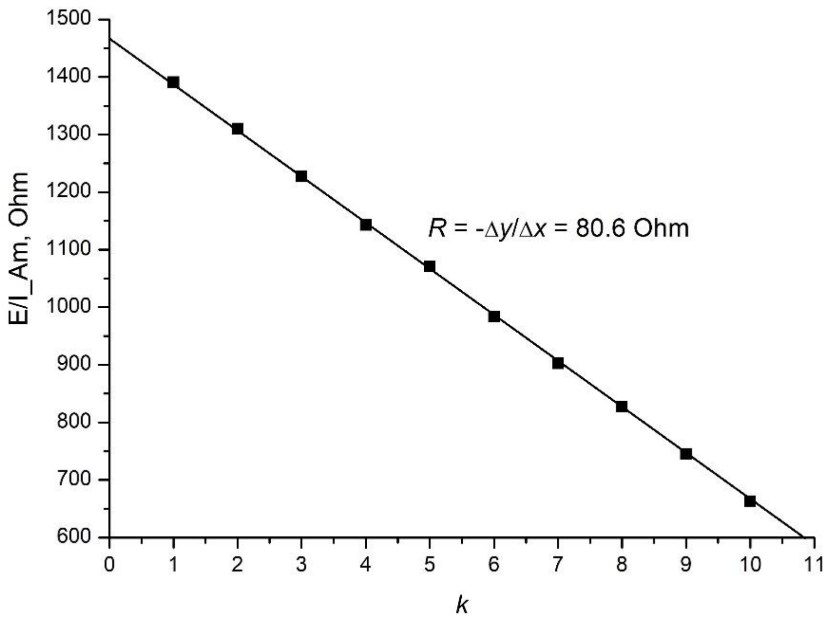
Using suitable variables represent the measurements on such a graph that the values of resistance R and number m can be obtained. Assume that $m \gg 1$ and $r < R$. Draw the dependence on the graph paper provided. All additionally calculated data must be present in the empty columns of the table. Calculate R and m. Round the value of R to a multiple of 5.

From (7) it follows that

$$\frac{E}{I_{\text{Am}}} = (N-m)R\left(1 + \frac{r}{mR}\right) + r = NR - mR + \frac{Nr}{m}. \qquad (8) \qquad \textbf{(1.0 p)}$$

As $m \gg 1$ and $r < R$, it is probable that $\frac{Nr}{m} \ll NR - mR$ and the dependence (8) can be nearly linear. **(1.0 p)**

Then from the slope of the graph of this dependence the resistance R can be determined. The column with the values of $\frac{E}{I_{\text{Am}}}$ is filled in the table. **(2.0 p)** We draw the graph $\frac{E}{I_{\text{Am}}} = f(k)$, where k are arbitrary consecutive whole numbers. **(5.0 p)**



Slope of the line is $-R$. From the graph $R \approx 80.6\ \Omega$. **(1.0 p)**

As the value must be multiple of 5, we round it to 80 Ω. **(1.0 p)**

Knowing already the value of R, the value of m can also be calculated. m can be calculated from each measurement from the formula $m = N - \frac{E}{I_{\text{Am}}R}$. We fill the column in the table. **(1.0 p)**

From the values follows that $m = 33$. **(1.0 p)**

c) (7 points) *Calculate r. Any additional calculated data should be present in the empty columns of the table. Round the value of r to a multiple of 5.*

From (8) follows that $r = \frac{m}{N}\left[\frac{E}{I_{\mathrm{A}m}} - (N-m)R\right]$. **(1.0 p)**

The value of r can be calculated from each measurement. We fill the column in the table. **(3.0 p)**

The average value of r is 19.8 Ω. **(1.5 p)**

As the value must be multiple of 5, we round it to 20 Ω. **(1.5 p)**