INSTITUT DES HAUTES ETUDES

POUR LE DEVELOPPEMENT DE LA CULTURE, DE LA SCIENCE ET DE LA TECHNOLOGIE EN BULGARIE

http ://balkanski-foundation.org/

Concours Général de Physique «Minko Balkanski»

31 juillet 2026

Прочетете внимателно!

Част първа и част втора съдържат условията на задачите съответно на френски и английски език. Единствените външни документи, на които имате право, са френски и английски речници. Употребата на калкулатори е позволена.

При оценяването на задачите голяма тежест ще имат **яснотата и стилът** на изложените решения и аргументация. Не използвайте излишна проза в аргументите си, бъдете кратки, точни и ясни. Използването на схеми за онагледяване на разсъжденията Ви е желателно. **Пишете само на езика, който сте избрали (френски или английски).**

Задачите носят еднакъв брой точки. Разпределяйте времето си разумно: ако се затрудните на даден въпрос или задача, можете спокойно да продължите нататък и да се върнете към него по-късно.

Разполагате с **4 часа**. Успех!

Part II

English

Problem I (10 points)

Phase-detection autofocus

This problem studies a simplified phase-detection autofocus system of the type used in single-lens reflex cameras. The main camera objective is modelled as a thin converging lens (L) with optical centre O, object and image foci F and F', and image focal length $f' = \overline{OF'} > 0$.

After unfolding the optical path produced by two redirecting mirrors, the autofocus module can be represented on the same axis as the objective. It contains a field lens (LC) of focal length $f'_c = 12\,\text{mm}$, two separator microlenses (LS_1) and (LS_2) of equal focal length $f'_s = 3\,\text{mm}$, and two linear sensors (C_1) and (C_2). The microlens centres are separated by $2a = 2\,\text{mm}$. Each sensor contains 48 equally spaced photodiodes along a 1 mm line; use the sampling pitch $d = 1\,\text{mm}/48$.

An electronic unit compares the two recorded intensity profiles. Their relative shift gives the sign and magnitude of the focusing error; a controller then drives a servomotor that translates the focusing group.

I. Formation of a sharp image

1. State the Gaussian conditions and approximations under which the thin lens (L) is used.

2. The object is represented by a segment (AB) perpendicular to the optical axis and its image by $(A'B')$. The point C is where the image sensor meets the axis. Define

$$p = \overline{OA}, \qquad p' = \overline{OA'}, \qquad x = \overline{CA}, \qquad x_0 = \overline{CO}, \qquad x' = \overline{CA'}.$$

Algebraic distances are positive from left to right, in the direction of light propagation; transverse segments are positive upwards and angles are positive counterclockwise.

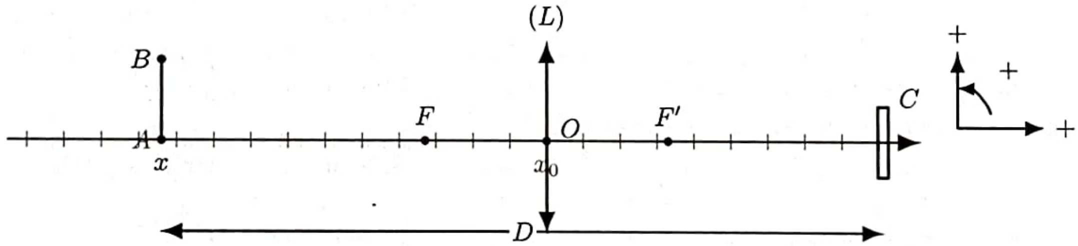


Figure I.1 – Image construction by the camera lens.

a) Starting from the Descartes thin-lens equation

$$\frac{1}{-p} + \frac{1}{p'} = \frac{1}{f'},$$

derive the transverse magnification relation connecting $\overline{AB}$, $\overline{A'B'}$, p and p', and hence obtain Newton's conjugation relation connecting $\overline{FA}$, $\overline{F'A'}$ and f'.

b) On the copy of the figure provided on the supplementary sheet, construct $(A'B')$ with a ruler using two principal rays. Check the position of A' using Newton's relation. Is the image formed on the sensor?

3. Let $D = |\overline{AC}|$ be the object-to-sensor distance and $T = \overline{OC} = -x_0 > 0$ the lens-to-sensor distance.

a) Assuming that the objective lies between the object and sensor, obtain the two possible solutions for T. Retain the solution $T < D/2$, corresponding to the objective close to the sensor, and show that

$$T = \frac{D - \sqrt{D^2 - 4Df'}}{2}.$$

b) Find the condition on D and f' for these positions to be real. Give a geometrical interpretation of the limiting case.

c) Determine T_∞ when the object is at infinity.

d) For $f' = 50\,\text{mm}$, the objective must focus between infinity and $D_{\text{min}} = 40\,\text{cm}$. Calculate the required travel $\delta = T(D_{\text{min}}) - T_\infty$. In which direction must the lens move as the object approaches?

e) A full-frame sensor of dimensions $36\,\text{mm} \times 24\,\text{mm}$ contains 24 million square pixels. The camera must resolve the 10 cm width of an object at 2.0 km; require its image to span at least two pixels. Neglecting diffraction and aberrations, determine the minimum focal length. The numerical data in this part are independent of part d).

II. Study of the autofocus module

After unfolding the optical path, the system is represented below. The point S is where the microlens plane meets the axis and P is where the photosensor plane meets it.

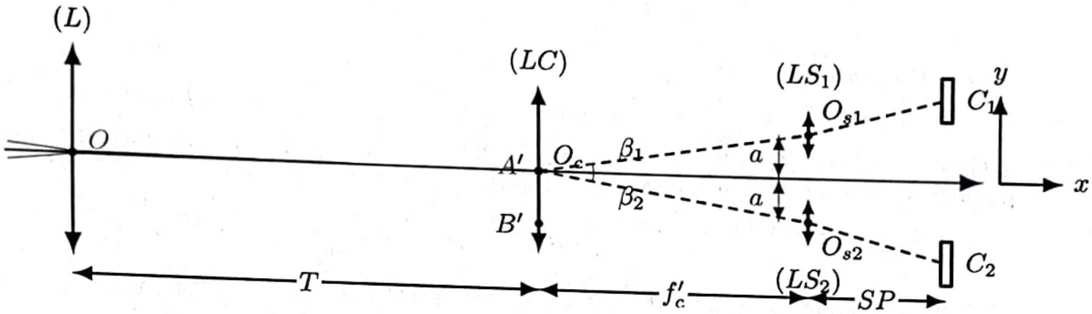


Figure I.2 – Unfolded model of the phase-detection autofocus module.

When focus is correct, the intermediate image $(A'B')$ is formed in the plane of the field lens (LC), whose optical centre is O_c. The microlenses lie in the image focal plane of (LC), so $O_cS = f'_c$.

4. First consider the correctly focused configuration.

a) The plane containing (LC) is conjugate to the photosensor plane through each microlens. Determine the distance SP.

b) On the corresponding figure on the supplementary sheet, complete the construction of the secondary images $(A''_1B''_1)$ and $(A''_2B''_2)$. The points A''_1 and A''_2, which are images of the axial point A', lie at the respective centres of (C_1) and (C_2). Compare the orientation of the secondary images with that of $(A'B')$.

c) Explain the role of the field lens: why does it allow both microlenses to receive bundles from the same point of the intermediate image? Which measured quantity carries the defocus information?

d) For the axial point A', use the geometry of the figure and the small-angle approximation to determine the two chief-ray angles β_1 and β_2 in terms of a and f'_c. State their signs, then calculate their magnitude.

5. The objective remains fixed and the initially focused object moves by a small distance $\Delta x > 0$ towards the camera. The primary image point then moves by $\Delta x'$ beyond the sensor plane. Use $f' = 50\,\text{mm}$ for numerical applications.

a) Starting from the thin-lens equation, show to first order that

$$\Delta x' = \left(\frac{T}{D - T}\right)^2 \Delta x.$$

For $D = 1.0\,\mathrm{m}$ and $\Delta x = 1.0\,\mathrm{cm}$, first calculate $\Delta x'$ using the exact value of T obtained in Question 3a), then give the estimate obtained from $D \gg T$ and $T \simeq f'$.

b) A', which lies to the right of (LC), is a virtual object for that lens. If A'_c is its image through (LC), show that

$$\overline{O_c A'_c} = \frac{f'_c\,\Delta x'}{f'_c + \Delta x'}.$$

Justify the approximation $\overline{O_c A'_c} \simeq \Delta x'$.

c) $\star$ Using the approximation from part b), let $s = f'_c - \Delta x'$ be the new object distance for the microlenses. Use their conjugation relation to show that both secondary images undergo the same longitudinal displacement

$$\Delta x''_1 = \Delta x''_2 = \left(\frac{f'_s}{f'_c - f'_s}\right)^2 \Delta x'.$$

d) $\star$ Using the rays through the microlens centres, show to first order that

$$\Delta\beta_1 = +\frac{a\,\Delta x'}{f'^2_c}, \qquad \Delta\beta_2 = -\frac{a\,\Delta x'}{f'^2_c}.$$

In the fixed photosensor plane, the centre of each profile follows the chief ray. Using the distance SP found in Question 4a), show that

$$\Delta y''_1 = SP\,\Delta\beta_1 = +\frac{a f'_s}{f'_c(f'_c - f'_s)}\Delta x', \qquad \Delta y''_2 = SP\,\Delta\beta_2 = -\frac{a f'_s}{f'_c(f'_c - f'_s)}\Delta x'.$$

Calculate $\Delta x''_1$, $\Delta x''_2$, $\Delta y''_1$ and $\Delta y''_2$ numerically.

e) Describe the relative shift of the two intensity profiles on the sensors and calculate its value. Compare it with the photodiode pitch d. Briefly explain why correlation of the complete intensity profiles may nevertheless measure a shift smaller than one pitch. How does the sign of the shift distinguish focus in front of the sensor from focus behind it?

6. The objective is now moved so that the image is formed again on the sensor. The algebraic lens displacement Δx_L is positive in the direction of light propagation.

a) What is the sign of Δx_L when the object has approached, so that $\Delta x > 0$?

b) Linearising the thin-lens equation while the object and lens move simultaneously, show that

$$\Delta x_L = K_L \Delta x, \qquad K_L = -\frac{T^2}{D(D - 2T)}.$$

For $D = 1.0\,\mathrm{m}$ and $\Delta x = 1.0\,\mathrm{cm}$, first calculate K_L and Δx_L using the exact value of T, then give the estimate obtained from $D \gg T$ and $T \simeq f'$.

7. Describe a complete autofocus loop: acquisition of the two profiles, estimation of their relative shift, determination of the error sign, motor command, and stopping criterion. Also give one situation in which phase-detection autofocus performs poorly.

Problem II (10 points)

How an Asteroid Forms Its Core

We study a fictitious spherical asteroid called *Balkania*. It is initially homogeneous: iron is uniformly dispersed in silicate material. A collision heats it and produces partial melting. Iron droplets may then sink and form a metallic core. The model is deliberately simplified; any additional assumption must be stated clearly.

Subparts marked ⋆ are more difficult. They may be left for a second pass; no later question depends on them unless the required result is supplied again.

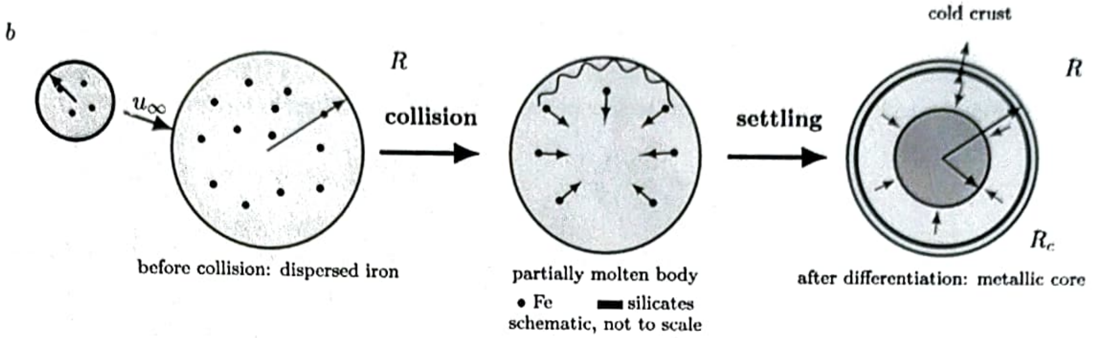


Figure II.1 – **Collision, partial melting, settling, and core formation.**

Data for Balkania.

Initial radius	$R = 2.0 \times 10^5$ m
Initial iron volume fraction	$\phi = 1/8$
Silicate density	$\rho_s = 3.0 \times 10^3$ kg m^{-3}
Iron density	$\rho_f = 7.0 \times 10^3$ kg m^{-3}
Mean specific heat capacity	$c = 1.0 \times 10^3$ J kg^{-1} K^{-1}
Initial temperature	$T_0 = 1250$ K
Onset-of-melting temperature	$T_s = 1400$ K
Effective latent heat of melting	$L = 2.5 \times 10^5$ J kg^{-1}
Thermal diffusivity	$\kappa = 1.0 \times 10^{-6}$ m^2 s^{-1}
Surface emissivity	$\varepsilon = 0.80$

Collision. The impactor is spherical, has radius $b = 8.0 \times 10^4$ m, and has the same composition and mean density as Balkania. Its speed at infinity is $u_\infty = 4.0 \times 10^3$ m s^{-1}. Balkania is treated as fixed during the approach. The collision is perfectly inelastic: the bodies remain joined. A fraction $\chi = 0.50$ of the kinetic-energy loss during the collision is converted into uniform internal energy of the merged body; mass loss is neglected.

Iron droplets. Their radius is $a = 5.0 \times 10^{-3}$ m. The dynamic viscosity of the molten silicate is $\eta = 20$ Pa s.

I. Impact heating and partial melting

1. Determine Balkania's mean density ρ. Then calculate the iron *mass* fraction.

2. Let M be the mass of Balkania and m the mass of the impactor.

a) Calculate M, m, and the ratio $\mu = m/M$.

b) Using conservation of the impactor's mechanical energy during the approach, show that

$$v_{\text{imp}}^2 = u_\infty^2 + \frac{2GM}{R+b},$$

where v_{imp} is its speed just before contact. Calculate v_{imp} and comment on the importance of gravitational acceleration.

3. Let V be the common speed of the two bodies immediately after the collision.

a) Using conservation of momentum, express V. Show that the kinetic-energy loss during the collision is

$$E_{\text{loss}} = \frac{1}{2}\frac{Mm}{M+m}v_{\text{imp}}^2.$$

b) Show that the deposited energy per unit mass of the merged body is

$$q_{\text{imp}} = \frac{\chi}{2}\frac{\mu}{(1+\mu)^2}v_{\text{imp}}^2,$$

and calculate it.

c) The energy $c(T_s - T_0)$ first heats the body to T_s. The remaining energy causes melting at approximately constant temperature. Determine the molten fraction x. Iron droplets can move when $x \geq x_c = 0.30$. Does the collision trigger differentiation?

d) $\star$Neglecting gravitational acceleration during the approach, determine the minimum impactor radius $b_{\text{min}} < R$ required to reach exactly $x = x_c$.

II. Sinking of iron droplets

In Parts II–IV, to keep the model analytically simple, use Balkania's initial values R and M and neglect the mass increase caused by the impactor. The gravitational field is approximated by that of the initial homogeneous sphere of density ρ. You may use the result that a homogeneous spherical shell produces no gravitational field at an interior point.

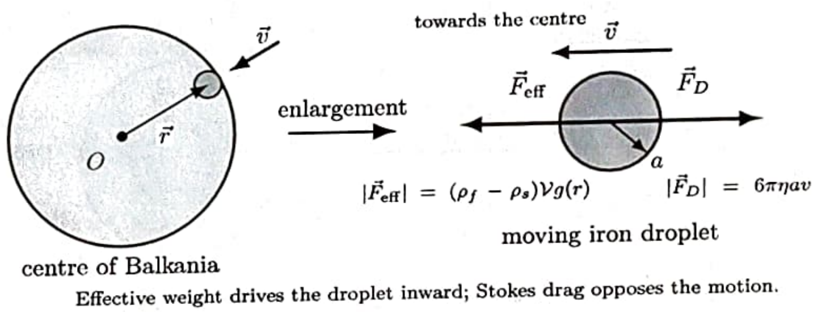


Figure II.2 – Position, velocity, and force balance of an iron droplet.

4. At a distance $r \leq R$ from the centre, show that the gravitational field strength is

$$g(r) = \frac{4\pi G\rho}{3}r.$$

Calculate the surface acceleration $g(R)$.

5. After differentiation, all the iron forms a pure spherical core of radius R_c. By conserving the total iron volume, determine R_c/R.

6. An iron droplet of volume $\mathcal{V}$ is at distance r from the centre.

a) Taking account of weight and buoyancy, show that the inward resultant force has magnitude

$$F = (\rho_f - \rho_s)\mathcal{V}g(r).$$

b) At low Reynolds number, the Stokes drag force has magnitude $F_D = 6\pi\eta a v$. Assuming terminal speed, show that

$$v_t(r) = \frac{2(\rho_f - \rho_s)a^2}{9\eta}g(r).$$

c) Calculate $v_t(R)$. Check the consistency of the Stokes regime using

$$\mathrm{Re} = \frac{2\rho_s a v_t}{\eta}.$$

7. ★ Over a very short time interval, take the position r and hence v_t as constant. Establish the equation of motion of a droplet released from rest in the form

$$\frac{dv}{dt} = \frac{v_t - v}{\tau}.$$

Express and calculate τ. Estimate the distance $v_t(R)\tau$ and justify the terminal-speed assumption used in Question 6.

8. Consider a droplet initially at the surface. Since $g(r) \propto r$, write $v_t(r) = kr$.

a) Obtain an expression for k, then calculate the settling time t_{set} from R to R_c.

b) Deduce the dependence of t_{set} on a and η. Determine the minimum droplet radius a_{100} that reaches the core in less than 100 yr, with all other data unchanged.

9. ★ At the end of differentiation, calculate the field $g_f(R_c)$ just outside the core, neglecting the field of the outer mantle. Compare it with the initial homogeneous field $g_i(R_c)$. Does the calculation of t_{set} tend to overestimate or underestimate the actual final settling time?

III. Energy released by differentiation

Assume radial ordering is preserved: the iron initially contained inside a sphere of radius r is found, after differentiation, inside a pure iron sphere of radius r_f.

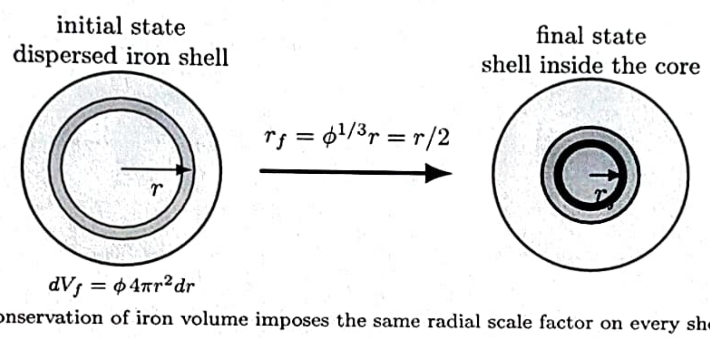


Figure II.3 – Radial mapping of iron shells during differentiation.

10. Show that $r_f = r/2$.

11. For a small iron volume initially at distance r, approximate its displacement as a motion from r to $r/2$ in the initial field.

a) Show that the gravitational work released per unit iron volume is

$$w(r) = (\rho_f - \rho_s)\int_{r/2}^{r} g(s)\,ds = \frac{\pi}{2} G\rho(\rho_f - \rho_s) r^2.$$

b) The iron volume initially contained in the shell $[r, r+dr]$ is $dV_f = \phi 4\pi r^2 dr$. Hence obtain the total released energy,

$$E_{\text{diff}} = \frac{\pi^2}{20} G\rho(\rho_f - \rho_s) R^5.$$

12. Assume that this energy is fully converted into heat within the asteroid.

a) Calculate the mean temperature rise $\Delta T_{\text{diff}} = E_{\text{diff}}/(Mc)$. Compare it with the impact heating.

b) Determine the radius $R_\star$ for which differentiation alone would produce a mean temperature rise of 100 K, with all other data unchanged. Interpret the result.

IV. Does the asteroid cool too quickly?

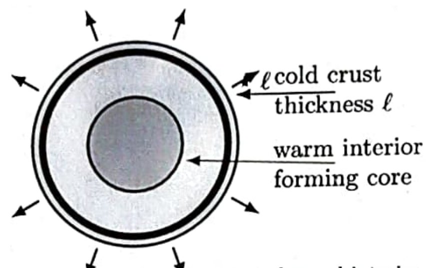
13. At temperature T_s, radiation received from space is neglected.

a) Calculate the initial radiated power

$$P_{\rm rad} = 4\pi R^2 \varepsilon \sigma T_s^4.$$

b) In an isothermal model, the molten fraction x first solidifies at $T \simeq T_s$. Taking the radiated power as constant and equal to its value in part a), estimate the time $t_{\rm sol}$ required to remove the latent heat MxL.

c) $\star$ After solidification, write the differential energy balance governing $T(t)$ when the radiated power varies as T^4. Using the mathematical hints, calculate the time $t_{\rm sens}$ required to cool from 1400 K to 1300 K. Estimate $t_{\rm iso} = t_{\rm sol} + t_{\rm sens}$ and compare it with $t_{\rm set}$.



Two limiting models are compared: an isothermal interior and purely diffusive transport.

Figure II.4 – Thermal structure, surface radiation, and diffusion depth.

14. In a solid or a liquid without efficient mixing, heat diffuses over a distance of order

$$\ell \simeq \sqrt{\kappa t}.$$

a) Calculate ℓ during $t_{\rm set}$ and compare it with R. Also estimate the diffusion time R^2/κ across the entire asteroid.

b) The exact volume fraction in the surface layer of thickness ℓ is

$$f_{\rm layer} = 1 - \left(1 - \frac{\ell}{R}\right)^3.$$

Estimate it, then explain how a cold crust can coexist with an interior that remains partially molten. Conclude whether the core can form within the diffusive model.

c) $\star$ How would very efficient internal convection modify this conclusion? A qualitative discussion is sufficient.

Annexe : constantes et relations utiles

Reference sheet : constants and useful relations

Constantes communes / Common constants

$$G = 6.67\times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}, \qquad \sigma = 5.67\times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}},$$

$$1\ \text{an} = 1\ \text{yr} = 3.16\times 10^{7}\ \text{s}, \qquad \ln 2 = 0.693, \qquad 1\ \text{rad} = 57.3^\circ.$$

Français
Les relations ci-dessous peuvent être utilisées directement. Elles ne remplacent pas le choix et la justification du modèle physique.

Problème I – petites variations et optique
Pour $|\theta| \ll 1$ rad :

$$\sin\theta \simeq \tan\theta \simeq \theta.$$

Au premier ordre :

$$F(z+\Delta z) - F(z) \simeq F'(z)\Delta z,$$

$$\Delta\left(\frac{1}{z}\right) \simeq -\frac{\Delta z}{z^2}.$$

Problème II – seuil, équations et intégrales
Pour la question 3d) :

$$q_c = c(T_s - T_0) + x_c L,$$

$$y = \frac{2q_c}{\chi u_\infty^2}, \qquad \frac{\mu}{(1+\mu)^2} = y,$$

$$y\mu^2 + (2y-1)\mu + y = 0.$$

La racine physique vérifie $0 < \mu < 1$.
Si $dv/dt = (v_t - v)/\tau$ et $v(0) = 0$:

$$v(t) = v_t\left(1 - e^{-t/\tau}\right).$$

Si $dr/dt = -kr$:

$$\ln(r_2/r_1) = -k(t_2 - t_1).$$

Intégrales utiles :

$$\int_{r/2}^{r} s\,ds = \frac{3r^2}{8}, \qquad \int_0^R r^4\,dr = \frac{R^5}{5},$$

$$\int T^{-4}\,dT = -\frac{1}{3T^3}.$$

Pour $|z| \ll 1$: $(1-z)^3 \simeq 1 - 3z$.

English
The relations below may be used directly. They do not replace the choice and justification of a physical model.

Problem I – small changes and optics
For $|\theta| \ll 1$ rad:

$$\sin\theta \simeq \tan\theta \simeq \theta.$$

To first order:

$$F(z+\Delta z) - F(z) \simeq F'(z)\Delta z,$$

$$\Delta\left(\frac{1}{z}\right) \simeq -\frac{\Delta z}{z^2}.$$

Problem II – threshold, equations, and integrals
For Question 3d):

$$q_c = c(T_s - T_0) + x_c L,$$

$$y = \frac{2q_c}{\chi u_\infty^2}, \qquad \frac{\mu}{(1+\mu)^2} = y,$$

$$y\mu^2 + (2y-1)\mu + y = 0.$$

The physical root satisfies $0 < \mu < 1$.
If $dv/dt = (v_t - v)/\tau$ and $v(0) = 0$:

$$v(t) = v_t\left(1 - e^{-t/\tau}\right).$$

If $dr/dt = -kr$:

$$\ln(r_2/r_1) = -k(t_2 - t_1).$$

Useful integrals:

$$\int_{r/2}^{r} s\,ds = \frac{3r^2}{8}, \qquad \int_0^R r^4\,dr = \frac{R^5}{5},$$

$$\int T^{-4}\,dT = -\frac{1}{3T^3}.$$

For $|z| \ll 1$: $(1-z)^3 \simeq 1 - 3z$.