Theoretical round. Sketches for solutions. <u>For jury job ONLY</u>

These solutions are written for jury members, so they contain a lot of explanations in the form of text. Students are instructed to use mostly formulas, calculations, drawings and graphs, writing texts is not allowed in the solutions. It is permitted to write a few keywords <u>in English</u> (for example, “Yes”, “No”, “Assume”, “Answer”, “Average”, “Minimum”, “Maximum”, “Impossible situation”, etc.), as well as terms contained in the accompanying tables and names of objects.

If you find typos, errors or missing points in the solution, please inform the organizing committee. Any suggestions for modifying the solutions are welcome, write justifications to gavrilovissp@list.ru and gavrilov@issp.ac.ru (to both addresses for reliability), if the suggestions are reasonable, the proposal can be approved.

1(αγ). Ancient observations.

1.1. The latitude may be found by the formula $\varphi = 90° – (h_1 + h_2)/2 = 24°11'$.
(Probably the observations were done on the territory of Murshidabad, the medieval city and capital of the former Gauda Kingdom).

1.2. The angle of inclination of the ecliptic to the plane of the celestial equator at that time was equal to:

$$\varepsilon = (h_1 - h_2)/2 = 23°38'.$$

1.3. The area that the northern circumpolar zone has lost is the ring between the Arctic Circle in 594 and the Arctic Circle in 2025, that is, between latitudes $\varphi_1 = 90° – 23°38' = 66°22'$ and $\varphi_2 = 90° – 23°26' = 66°34'$. If R is the radius of the Earth, φ is expressed in radians, and $\varphi_0 = 66°28'$ is the mean value of φ, then the area of this ring is:

$$S = R\cos\varphi_0 \cdot R(\varphi_1 - \varphi_2) = 23°38'.$$

$$S = (6371\ \text{km})^2 \cdot 0.399\cdot(12' / 3438'/\text{rad}) \approx 56\ 500\ \text{sq.km}.$$

The answer is: –56 500 sq.km.

2(αγ). NGC of the Year.

NGC of the Year. The altitude H, which must be reached to see an object located slightly below the horizon (at a small angle η), is determined by the formula, which is easy to obtain from the properties of a right-angled triangle by drawing the appropriate drawing:

$$(R+H)^2 = R^2 + (\eta R)^2 \quad\Rightarrow\quad H \approx \eta^2 R/2,$$

where R is the radius of the Earth, and the angle η must be taken in radians. We can substitute the value of the radius of the Earth and convert radians to degrees, then we get the formula with a coefficient 970 m, connecting H (in meters) with η (in degrees):

$$H(m) \approx 970\,m \cdot \eta(degrees)^2,$$

To see the whole cluster, you need to see its lowest point. The upper culmination of the lowest point of NGC 2025 in Dhaka will be at an altitude of

$$h_0 = 90° - \varphi + \delta - \varrho/2 = 90° - 23°48' - 71°43' - 1' = -5°32',$$

that is, $5°32'$ below the mathematical horizon. To compensate for this angle, you need to rise up to an altitude of about 30 km. This is an unimaginable height, exceeding the record for rising up in a thermal balloon.

But have we considered everything? Recall that there is refraction at the horizon, $\rho \approx 35'$, which “raises” objects a little. Taking this refraction into account

$$h_1 = h_0 + \rho = -4°57',$$

In this case, you need to rise up about 24 km. Now let's note that when observing from such an altitude, refraction works almost twice: the first time on the way from the celestial object to the “point of contact of the ray of vision with the surface of the Earth”, and the second time on the way from the “point of contact” to the observer.

$$h_2 = h_0 + 2\rho = -4°22',$$

In this case, you need to reach about 18.5 km altitude. This is already an achievable height for a thermal balloon. This answer is an estimation.

You can go deeper and calculate the refractive angle more accurately. Using the formula for the dependence of atmospheric pressure on altitude, it can be calculated that at an altitude of about 18.5-20 km, atmospheric pressure is approximately 10% of the pressure at sea level. This means that 90% of the atmosphere is located below, respectively, for the second time, in the area from the “touch point” to the observer, refraction must be taken into account by a factor of 0.9.

$$h_3 = h_0 + 1{,}9\rho \approx -4°25',$$

Taking this amendment into account, it is necessary to rise up about 19 km.

3(αβγ). Comet Lemmon. If the comet was observed in the second half of the 7th century AD during its previous approach to the Sun, then its orbital period in the previous period can be estimated as 1350±25 years. According to Kepler's Third Law, such a period corresponds to the semimajor axis of the orbit

$$a = T^{2/3} = 122 \pm 1.5 \text{ au}.$$

The eccentricity of the orbit:

$$e = (a - p)/a = 0.99566 \pm 0.00005.$$

3.1. Such a small difference in eccentricity from unity indicates that the comet is moving along an almost parabolic trajectory. The velocity of the comet at perihelion (i.e., the speed at the moment) is almost equal to the II cosmic one:

$$V_{II} = \sqrt{2GM_\odot/p} = \sqrt{2 \cdot 6.67 \cdot 10^{-11} \cdot 1.99 \cdot 10^{30}/(0.53 \cdot 1.496 \cdot 10^{11})} = 5.79 \cdot 10^4 \ m/s = 57.9 \ km/s.$$

To calculate, we converted astronomical units to meters. The result is a value in meters per second. Thus, the comet's perihelion velocity

$$V_p \approx V_{II} = 57.9 \ km/s \approx 58 \ km/s.$$

3.1*. A more accurate result can be obtained by applying the law of conservation of energy and Kepler's Second Law for the points of perihelion and aphelion of the orbit of the comet:

$$\frac{V_p^2}{2} - \frac{GM_\odot}{p} = \frac{V_a^2}{2} - \frac{GM_\odot}{a}, \qquad V_p \cdot p = V_a \cdot a,$$

where G is the gravitational constant, $M_\odot$ is the mass of the Sun, p and a the distances to the Sun at perihelion and aphelion (which are equal to the radii of the orbits of the Earth and Saturn) and V_p and V_a are the corresponding velocities.

Solving these two equations together, we get:

$$V_p = \sqrt{\frac{2GM_\odot}{p(p/a + 1)}} = \sqrt{\frac{2GM_\odot}{p((1-e)+1)}} \approx 57.8 \ km/s.$$

As you can see, the answer is only 0.2% different from the one obtained for the parabolic velocity and is completely unnecessary for the estimation solution. However, this solution is absolutely correct.

A note to the jury. The part of the text highlighted in blue (p.1.1.) is not required for the solution of p.1.1., but may be useful for one of alternatives of the solutions to p.1.2.*

3.2. But in this part of the problem, high accuracy is required, which was not important in clause 1.1. Using the formula obtained earlier

$$V_p = \sqrt{\frac{2GM_\odot}{p(p/a+1)}} = \sqrt{\frac{2GM_\odot}{p(2-e)}},$$

we can get

$$2-e = \frac{2GM_\odot}{pV_p^2},$$

for small variations

$$\frac{\Delta(2-e)}{2-e} = \Delta\left(\frac{2GM_\odot}{pV_p^2}\right)\Big/\frac{2GM_\odot}{pV_p^2},$$

$$\frac{-\Delta e}{2-e} = \Delta\left(\frac{1}{V_p^2}\right)\Big/\frac{1}{V_p^2} = \frac{-2\Delta V_p}{V_p},$$

since Δe is significantly less than $(2-e)$, and $(2-e) \approx 1$, we obtain

$$\Delta e \approx \frac{2\Delta V_p}{V_p} = -2\delta \approx -0.046\,\%\,.$$

The new eccentricity is:

$$e_1 = (a_1 - p)/a_1 = e\cdot(1 - 0{,}00046) = 0{,}9952, \quad 1 - e_1 = 0{,}0048.$$

The new semi-major axis of the orbit of the comet:

$$a_1 = p/(1-e_1) = 0{,}5299\text{ au}/0{,}0048 \approx 110\text{ au}.$$

The new orbital period:

$$T_1 = a_1^{3/2} \approx 1160\,years.$$

$2025 + 1160 = 3185$. Thus, the next time inhabitants of the Earth will see comet C/2025 A6 (Lemmon) will be at the end of the 32th century (*). This is two centuries earlier than if the comet had not lost 0.023% of its velocity at perihelion.

(*) Provided that there are no major changes in the orbit in the outer part of the trajectory of the comet. There, even a minor affect from massive objects can lead to a significant change in trajectory, up to ejection of the comet outside the Solar System.

4 (αβγ). On a remote exoplanet. To determine the spectral class of the star Erythrelatis, it is necessary to find the approximate temperature of its photosphere.

The thermal balance of the planet is determined by the incoming and outgoing energy per unit of time.

$$W_{in} = W_{out}\,.$$

Incoming power:

$$W_{in} = (1-\alpha)\cdot\sigma T_{star}^4\cdot\frac{\pi(\beta/2)^2}{4\pi}\cdot 4\pi R_{planet}^2 = (1-\alpha)\cdot\sigma T_{star}^4\cdot\frac{\pi\beta^2}{4}\cdot R_{planet}^2\,,$$

where σ is the Stefan–Boltzmann constant, α is the albedo of the planet, and β is the angular diameter of the central star visible from the planet.

Outgoing (radiated) power:

$$W_{out} = \sigma T_{planet}^4\cdot 4\pi R_{planet}^2 = \sigma(T_{obs} - T_{GHE})^4\cdot R_{planet}^2\,,$$

where T_{planet} is the effective temperature of the planet, T_{obs} is the observed mean temperature of the planet, and T_{GHE} is the greenhouse effect.

From these two equations:

$$(1-\alpha)\cdot T_{star}^4\cdot\frac{\pi\beta^2}{4}=(T_{obs}-T_{GHE})^4\,,$$

From where:

$$T_{star}=(T_{obs}-T_{GHE})\cdot\sqrt[4]{\frac{4}{\pi\beta^2(1-\alpha)}}\,.$$

Coefficients are not needed to compare the Sun and the star Erythrelatis, therefore:

$$T_{star}\sim(T_{obs}-T_{GHE})\cdot\beta^{-1/2}(1-\alpha)^{-1/4}\,,$$

$$T_{ery}\,=\,T_{sun}\cdot\frac{(T_{obs}-T_{GHE})_{ant}}{(T_{obs}-T_{GHE})_{earth}}\cdot\left(\frac{\beta_{ery}}{\beta_{sun}}\right)^{-\frac{1}{2}}\cdot\left(\frac{1-\alpha_{ant}}{1-\alpha_{earth}}\right)^{-\frac{1}{4}}.$$

The angular diameter of the star Erythrelatis, visible from the planet Antechea, can be found by analyzing the data of the length of day and the duration from rise to set of the central star given in the table. For the Earth, we see that the time from sunrise to sunset is 406 seconds longer than half a day. Converting this value to angular minutes (the length of day corresponds to 21600 angular minutes), we get δ_{earth} = 101.5′. It can be verified that this addition is caused by refraction (twice ρ_{earth} = 34.8′ at sunrise and sunset, data from the table of constants) and the angular diameter of the Sun (twice the angular radius of the Sun: at sunrise and sunset, β_{sun} = 31.9′ in total).

For Antechea, the time from rise to set of Erythrelatis is 324 seconds longer than half a day. Converting this value to angular minutes (note that the conversion coefficient is different here: 21 600 angular minutes corresponds to 70 832 seconds), we get δ_{ant} = 98.8′. This is also the doubled refraction at the horizon plus the apparent angular diameter of Erythrelatis.

A significant error is possible here: twice subtract ρ_{earth} = 34.8′ from δ_{ant} = 98.8′, and get β_{ery} = 29.2′. (Looking ahead, we note that in this highly erroneous case, T_{ery} ≈ 6300 K and spectral class F8 are obtained.)

The value of refraction is proportional to the density of the atmosphere at the surface of the planet, which is the same as the ratio of pressure to temperature. (That this is the relationship can also be understood from the refraction formula given in the table of constants.) The temperature near the surface on both planets is the same, so the value of refraction is proportional to the atmospheric pressure:

$$\rho_{ant}\,=\,\rho_{earth}\cdot\frac{P_{ant}}{P_{earth}}=15.08'\,.$$

Thus, the angular diameter of the star Erythrelatis, as seen from the planet Antechea, is:

$$\beta_{ery}\,=\,\delta_{ant}-2\rho_{ant}\approx 68.6'\,.$$

According to the formula obtained above for T_{ery} :

$$T_{ery}\,=\,5777K\cdot\frac{259K}{254K}\cdot\left(\frac{68.6'}{31.9'}\right)^{-\frac{1}{2}}\cdot\left(\frac{0.64}{0.71}\right)^{-\frac{1}{4}}=4123K\approx 4100K\,.$$

Using the Hertzsprung-Russell diagram, we determine that stars of the K7 class have such a temperature.

The answer to the problem does not depend on neither the equatorial radius of the planet [#1], the mass of the planet [#2] nor the acceleration of gravity [#3].

Answer: the spectral class of the star Erythrelatis is **K7**. Not depending on [#1, 2, 3]

5(αβγ). Canopus and Sirius.

5.1. Canopus is visible in Dhaka, but not high above the horizon. Therefore, the best conditions for the observation should be at the time of its upper culmination. The upper culmination of a celestial body occurs when its sidereal time equals to its right ascension. It is $6^h\,24^m$ for Canopus. The sidereal time is equal to the mean solar time at the date of autumnal equinox, and then begins to outstrip it with speed $1^d/365.2422$ per day (about $3^m\,56^s$ per day). 47-48 days passed since September 22 (autumnal equinox) till November 8-9 (the next 24 hours), let us use 47.5 days difference. The difference between the mean solar time and the sidereal time is now:

$$24^h\,00^m \times 47.5/365.2422 = 3^h\,07^m$$

Thus, now sidereal time $6^h\,24^m$ corresponds to

$$6^h\,24^m - 3^h\,07^m = 3^h\,17^m \quad \text{mean solar time.}$$

Dhaka is located about 0.5° East from the 90° meridian, so the mean solar time here is $0.5° \times 4^{m/°} = 2^m$ ahead of the local time UT+6 used in Bangladesh. So to get the answer for UT+6, you need to subtract these 2^m, and we get

$$3^h\,17^m - 0^h\,02^m = 3^h\,15^m \quad \text{time of Bangladesh.}$$

Thus, there will be favorable conditions for observing Canopus tonight. Don't miss the chance!

5.2. During the upper culmination, Canopus is at an altitude

$$h_c = 90° + \delta_c - \varphi = 90° - 52°42' - 23°48' = 13°30'\ .$$

At this time, Sirius is also located near the upper culmination (from the difference in right ascensions, it can be seen that its culmination will be only 21 minutes later), so its altitude is approximately equal to

$$h_s \approx 90° + \delta_s - \varphi = 90° - 16°43' - 23°48' = 49°29'\ .$$

If the light of celestial bodies from the zenith passes through a layer d_0 of polluted air, then the layer from the stars at an altitude of h: $d_1 = d_0/sinh$.
For Canopus, the layer is $d_c = d_0/sin13°30' = 4.28\, h_0$,
And for Sirius the layer is $d_s = d_0/sin49°29' = 1.32\, h_0$.
Under the most favorable conditions, a layer of one atmosphere weakens light by 0.23^m, thus, the extratmospheric magnitude of Canopus is

$$-0.72^m - 0.23^m = -0.95^m.$$

We will assume that the limit of visibility by the naked eye is 6^m. Thus, the layer $d_c = 4.28\, h_0$ weakens the light by $6^m - (-0.95^m) = 6.95^m$. Accordingly, the layer d_0 weakens the light by $6.95^m/4.28 = 1.62^m$, and a layer $d_s = 1.32\, h_0$ weakens the light by $1.62^m \cdot 1.32 = 2.13^m$.
The extratmospheric magnitude of Sirius is

$$-1.46^m - 0.23^m = -1.69^m.$$

Thus, in the conditions of smog of the considered level, the apparent magnitude of Sirius will be

$$-1.69^m + 2.13^m = 0.44^m.$$

6(βγ). Model of the Universe.

6.1. In this model, the Sun makes 1 revolution around the Earth (more precisely, around the Earth—Moon system) relative to the sphere of fixed stars in 1 year. According to Kepler's Third Law, comparing this motion with the orbital motion of the Earth–Moon system:

$$\frac{T_{sun}^2}{R_{sun}^3} = \frac{T_{moon}^2}{R_{moon}^3} \quad \Rightarrow \quad R_{sun} = R_{moon} \cdot \left(\frac{T_{sun}}{T_{moon}}\right)^{2/3}$$

Thus, the distance to the Sun in this model is:

$$\mathrm{R_{sun}} = 384400\ km \cdot \left(\frac{365.256\ day}{27.3217\ day}\right)^{2/3} = 2.165\ million\ km.$$

The angular size of the Sun is 32′. The physical diameter of the Sun in this model is:

$$D_{sun} = 2{,}165\, million\; km \cdot \left(\frac{32'}{3438'}\right) = 20150\; km \approx 20000\; km.$$

6.2. In this model, the base for measuring angles could only be the diameter of the Earth. If the accuracy of measuring angles in the sky at that time was about δ = 10″, then the distances could only be measured to those objects from which the diameter of the Earth is visible at an angle greater than this value, in other words, to those objects whose doubled daily parallax is greater than δ = 10″. That is, the distance up to

$$L = \frac{D_{earth}}{\delta} = \frac{12742\text{ км} \cdot 206265''}{10''} \approx 260\; million\; km.$$

This means that the sphere of fixed stars can be located at any distance greater than 260 million km.
In our real Universe, this is 1.8 au, that is, the characteristic distances of the inner part of the Solar System (the order of the distances to the Sun, Mercury, Venus and Mars).
Answer: **> 260 million km**.

7(βγ). Distant galaxy. From the formula

$$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

it is easy to understand that the first line (that is, the line with the longest wavelength) in each series is the line for which $n = m + 1$. Therefore, for the ratio between the inverse wavelengths for H**α** and the first line of the Brackett series, we can write:

$$\frac{1}{\lambda_\alpha} \Big/ \frac{1}{\lambda_B} = \left(\frac{1}{2^2} - \frac{1}{3^2}\right) \Big/ \left(\frac{1}{4^2} - \frac{1}{5^2}\right),$$

$$\lambda_B/\lambda_\alpha = \left(\frac{9-4}{36}\right) \Big/ \left(\frac{25-16}{400}\right) = \frac{500}{81}.$$

The wavelength λ_B itself may not even be calculated, as it is possible not to know the wavelength λ_α. The redshift for this galaxy is

$$z = \left(\frac{\lambda}{\lambda_0} - 1\right) = \frac{419}{81} \approx 5{,}17\,,$$

If we use the Doppler formula from classical mechanics ($v/c = (\lambda - \lambda_0)/\lambda_0$), we get the galaxy receding with a velocity of z·c = 5.17 c, greater than the speed of light. Therefore, since the redshift is large, we need to use the formulas of special relativity for an accurate answer. But to solve the problem as formulated (with an accuracy of 10%), it is not necessary to know these formulas, it is enough to realize that the relative velocity of the galaxy is close to the speed of light, i.e. it is moving away from us at a speed of about 300000 km/s (answer to 7.1.), and the distance to it can be found from the Hubble formula v = H·R. It is about

R = c / H = 300 000 km/s / 68 km/s/Mpc ≈ 4400 Mpc ≈ 4.4 Gpc (answer to 7.2.).

Note. The exact solution (not required by the condition of the problem):
Redshift $z = (\lambda - \lambda_0)/\lambda_0 = (40525\text{ Å} - 6563\text{ Å}) / 6563\text{ Å} = 5.17.$

$$z + 1 = (1 + v/c) / (1 - v^2/c^2)^{1/2}$$

$$v/c = [(z+1)^2 - 1] / [(z+1)^2 + 1]$$

Then for z = 5.17 obtain v = 0.949·c ≈ 285 000 km/s (answer to 7.1.).
According to the Hubble formula (v = H·R), we obtain

R = v / H = 285 000 km/s / 68 km/s/Mpc ≈ 4190 Mpc ≈ 4.2 Gpc (answer to 7.2.).