Theoretical round. Sketches for solutions. <u>For jury job ONLY</u>

These solutions are written for jury members, so they contain a lot of explanations in the form of text. Students are instructed to minimize text, mostly use formulas, calculations, drawings and graphs, writing extra texts is not encouraged in the solutions. It is permitted to write a few keywords <u>in English</u> (for example, "Yes", "No", "Assume", "Answer", "Average", "Minimum", "Maximum", "Impossible situation", etc.), as well as terms contained in the accompanying tables and names of objects.

The given sketches are not full; the team leaders have to give more detailed explanations to students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

Jury members should evaluate the student's solutions in essence, and not by looking on formal existence the mentioned sentences or formulae. The formal presence of the mentioned positions in the text is not necessary to give the respective points. Points should be done if the following steps de facto using these positions.

1(αγ). Crab nebula. The distance to an object having a parallax of 0.0005" (this value can be found in the table of stars and objects) is equal to:

$$L = \frac{1}{p} = \frac{1}{0.0005} = 2000\, pc \approx 6500\, ly.$$ 1p.

The apparent angular size of 5' at this distance corresponds to the physical size

$$2000\, pc \times \frac{5'}{3438'} = 2{,}9\, pc \approx 3\, pc \quad \text{или} \quad 6500\, ly \times \frac{5'}{3438'} \approx 9.5\, ly,$$

which is

$$2{,}9\, pc \times 206265 \frac{au}{pc} \approx 6 \cdot 10^5\, au \approx 9 \cdot 10^{13}\, km \quad or \quad 9.5 \times 365 \times 86400 \times 300000 \approx 9 \cdot 10^{13}\, km.$$ 1p.

However, this is the size of the nebula at the time when the light from it began to come to us, that is, 6500 years ago. Thus, the total age of the Crab Nebula is currently 969 + 6500 ≈ 7500 years. And if in 969 years it expanded to $6 \cdot 10^5$ au or $9 \cdot 10^{13}$ km, then, assuming a linear expansion, in 7500 years it expanded by

$$6 \cdot 10^5\, au \times \frac{7500\, year}{969\, year} = 4{,}6 \cdot 10^6\, au \approx 5 \cdot 10^6\, au \quad \text{или} \quad 9 \cdot 10^{13}\, km \times \frac{7500\, year}{969\, year} \approx 7 \cdot 10^{14}\, km.$$ 3p

Or, calculating differently, in astronomical units it will be

$$9 \cdot 10^{13}\, km \times \frac{7500\, year}{969\, year} \approx 7 \cdot 10^{14}\, km \Big/ 150000000 \frac{km}{au} \approx 4.7 \cdot 10^6\, au.$$ 1p

Answer: $7 \cdot 10^{14}$ km, $5 \cdot 10^6$ au Accuracy with more than 1 significant digit is inappropriate. 2p

2(αγ). Space apparatus. From the condition

$$m\frac{V^2}{R+h} = \frac{GMm}{(R+h)^2}$$

we get that the speed of the apparatus is equal to

$$V = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{6.674 \cdot 10^{-11} \times 5.97 \cdot 10^{24}}{6.37 \cdot 10^6 + 0.32 \cdot 10^6}} = 7.72 \frac{\text{km}}{\text{s}}.$$

During this time, the apparatus overcomes the distance

$$S = Vt = 7.72\frac{km}{s} \times 110\,s \approx 850\,km,$$

half of which it is visible before passing the zenith point, and half after. At such distances, we can neglect the curvature of the Earth and consider motion in a flat approximation. Thus, the satellite is visible at altitudes above

$$\beta = arcctg\left(\frac{s/2}{h}\right) = arcctg\left(\frac{425\,km}{320km}\right) = 37°.$$

If from the zenith point the light from the device passes a layer of polluted air h_0, then from a point at an altitude of 37° it passes a layer

$$h_1 = {}^{h_0}/_{sin\beta} = 1.66\,h_0$$

We will assume that the limit of visibility with the naked eye is 6^m. Thus, an additional layer $h_1 - h_0 = 0.66\,h_0$ weakens the light by $6^m - 1.5^m = 4.5^m$. Accordingly, the h_0 layer will weaken the light by $4.5^m/0.66 = 6.8^m$. So we find that extra-atmospheric apparent magnitude of the apparatus is equal to $1.5^m - 6.8^m = -5.3^m$. Under the most favorable conditions, a layer of one atmosphere weakens the light by 0.23^m. Thus, at the zenith, under the most favorable conditions, the device would be visible as an object of $-5{,}3^m + 0{,}23^m \approx 5^m$ magnitude.

3(αβγ). Interstellar Travel.

3.1. Distance between the Sun and α Cen:

$$\mathrm{r} = \frac{1}{\pi} = \frac{1}{0.747} = 1.34\,pc = 4.36\,ly = 276\,000\,au$$

(the parallax of α Cen can be found in the table of stars and objects).

Flux of the Sun seen at α Cen:

$$\mathrm{F}_{\odot} = 1367\,{}^{W}/_{m^2} \times \left(\frac{1au}{r}\right)^2$$

(the value of the solar constant can be found in the table of constants).

Solid angle of the engine plume:

$$s = \pi\frac{(4.96/2)^2}{\left(\frac{180}{\pi}\right)^2} = 5.89 \cdot 10^{-3}$$

Flux from the ship at a distance of r_s:

$$\mathrm{F}_s = \frac{2P \times 0.0992}{sr_s^2}$$

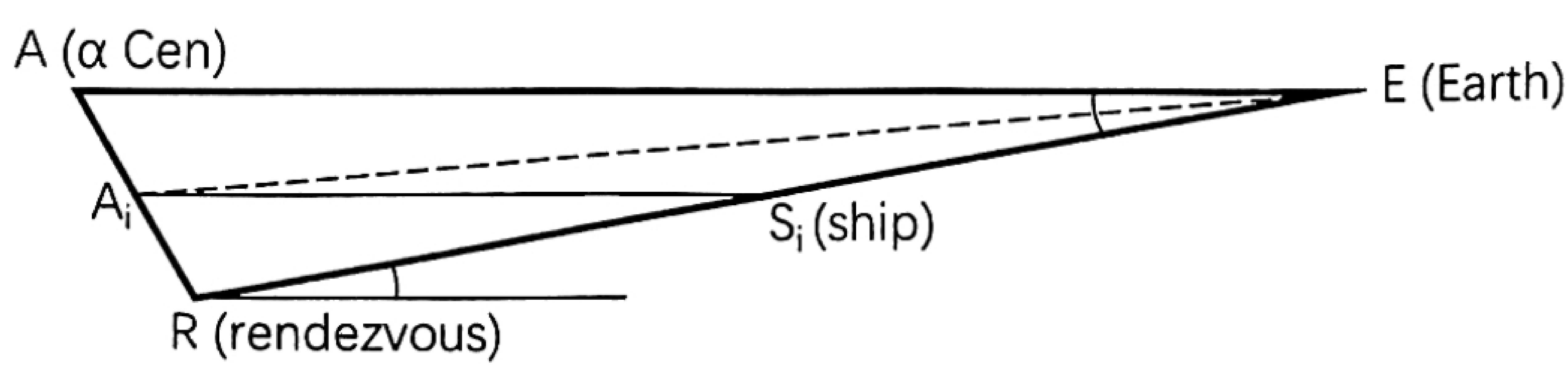
When the two fluxes equals, $\mathrm{F}_{\odot} = \mathrm{F}_s$.

$$1367\,{}^{W}/_{m^2} \times \left(\frac{1au}{r}\right)^2 = \frac{2P \times 0.0992}{5.89 \cdot 10^{-3} \cdot r_s^2} \quad \Rightarrow \quad r_s = \sqrt{\frac{4.5 \cdot 10^{17}\,\mathrm{W} \times 0.0992 \times (276\,000\,au/1au)^2}{1367\,{}^{W}/_{m^2} \times 5.89 \cdot 10^{-3}}}$$

$$\mathrm{r_s} = 2.06 \cdot 10^{13}\,m = 137\,au.$$

3.2. Considering both the ship and Alpha Centauri move at constant speed, the relative direction of them (A_iS_i) remains the same to the departure positions (AE). So seen at Alpha Centauri, the angular distance of the ship and the sun ($\angle EA_iS$) increases, and reaches maximum just before the arrival. At the arrival, the angular distance is equal to ∠AER, which is actually the angular movement of Alpha Centauri during the transit time. Since the distance of Alpha Centauri does not change much during the years, the angular movement can easily be estimated by proper motion multiplied by duration.

There is no aberration since there is no apparent motion.



TLDR: $A_iS_i \parallel AE$, max(∠EAS)=∠AER=proper motion × duration

Total proper motion:

$$\mu = \sqrt{3.614''^2 + 0.803''^2} = 3.7''/yr$$

Ship transit time:

$$t = \frac{r}{v} = \frac{4.36\ ly}{0.152\ c} = 28.6\ yr$$

Maximum angle:

$$\angle REA = 28.6\ \text{yr} \times 3.7''/\text{yr} = 106''. \quad 106'' > 1'$$

It's greater than the naked eye resolve angle of common practice.

Conclusion: "**Yes**".

4(αβγ). Hybrid eclipse. (Drawing is desirable). At points 1 and 3, where the annular eclipse changed to the total and the total to the annular, the apparent angular dimensions of the Moon and the Sun coincide. These points are on the tip of the cone of the total eclipse. Compared to the center of the Earth, these points are closer to the Moon by

$$L_1 = R_E \cdot \sin 10° = 6370\ \text{km} \times 0.173 \approx 1100\ \text{km}$$

(here we took into account the almost central passage of the cone around the globe, the cosine of the corresponding angle of deviation is almost equal to 1). Point 2, where a total eclipse with a maximum phase is observed, is closer to the Moon (also compared to the center of the Earth) by

$$L_2 = R_E \cdot \sin 67° = 6370\ \text{km} \times 0{,}92 \approx 5850\ \text{km}.$$

The difference between these distances is

$$\Delta L = L_1 - L_2 = R_E \cdot (\sin 67° - \sin 10°) \approx 4750\ \text{km}.$$

The width of the cone of the Moon's shadow at this distance is

$$D = \Delta L \cdot \beta = 4750\ \text{km} \times 32'/3438' \approx 44\ \text{km}.$$

The axis of the cone come to the Earth's surface at an angle of 67° to the normal and almost perpendicular to the trajectory of the shadow, so to get the width of the path, you just need to divide the diameter of the cone by the sine of the angle of the axis of the cone to the normal.

$$W = D / \sin 67° \approx 44\ \text{km} / 0{,}92 \approx 48\ \text{km},$$

and, taking into account a low accuracy of our calculations,

$$\approx 50\ \text{km}.$$

Note. In reality, the maximum width of the path of this totality was 49 km.

5(αβγ). Meteors. 3p - probability approach

5.1. Denote by N_0 the total number of meteor and by N_1 and N_2 the number of meteors seen by the first and second observers, respectively, and by N_{12} – the number of meteors seen by the both observers. The first observer of the total number of meteors N_0 noticed the fraction $\nu_1 = \frac{N_1}{N_0}$ (the so called coefficient of noticability), and the second – the fraction $\nu_2 = \frac{N_2}{N_0}$. Of the meteors seen by the second observer, the first one noticed exactly the same fraction of ν_1, that is, $\nu_1 = \frac{N_{12}}{N_2}$. Hence, $\frac{N_{12}}{N_2} = \frac{N_1}{N_0}$. And so $N_0 = \frac{N_1 N_2}{N_{12}}$. 1,5p

Thus, the total number of meteors is $N_0 = \frac{28 \cdot 38}{18} \approx 59$. 1,5

5.2. An artistic picture. 0,5+1,5

6(βγ). Dyson sphere.

6.1. The radius of the Dyson sphere R_D is found from the thermal equilibrium condition. All the energy coming from the Sun (R_S = 696 000 km, T_S = 5777 K) will eventually be re-emitted by the sphere.

The temperature on the sphere should correspond to comfortable conditions for earthlings. Take T_D = 293 K.

$$\sigma T_S^4 \cdot 4\pi R_S^2 = \sigma T_D^4 \cdot 4\pi R_D^2 \quad \Rightarrow \quad R_D = R_S \frac{T_S^2}{T_D^2} = 271 \text{ млн. км} \approx 1{,}8 \text{ а. е.}$$

6.2. If the construction of the sphere took all the substance of the Earth, then equate their volumes:

$$4\pi R_D^2 d_D = {}^4\!/_3\, \pi R_E^3 \quad \Rightarrow \quad d_D = \frac{R_E^3}{3R_D^2} = 1.2 \text{ мм.}$$

A very thin sphere.

6.3.

$$g_D = GM/R_D^2 \approx 1.8\ {}^{\text{мм}}\!/_{\text{с}^2}$$

Asteroid (minor planet).

6.4. In order for day and night to alternate, as on Earth, the hemisphere must orbit the Sun with a period T = 1 сутки = 24 часа = 86400 с. Let's find the force of attraction of the hemisphere to the Sun. Let's denote the plane bounding the hemisphere as XZ, and the axis perpendicular to it as Y. Obviously, the total force of attraction will be directed along the Y axis. Consider the attraction of a single section of the hemisphere with mass $\Delta m = \rho_F d_F \Delta S$. In the projection on the Y axis , the force of attraction will be equal to

$$\Delta F = \frac{GM\Delta m}{R_F^2} \cos\varphi,$$

where φ is the angle from the Sun between the direction to this unit area and the Y axis. At the same time, the projection of this section onto the XZ plane is also proportional to $\cos\varphi$. It turns out that the sections having the same projections on the XZ axis give the same contribution to the force of attraction. As a result, if the entire mass of the hemisphere is proportional to the area of the hemisphere $2\pi R_F^2$ and is equal to $m = \rho_F d_F \cdot 2\pi R_F^2$, then the total force of attraction is proportional to the projection area of πR_F^2 and is equals to

$$F = \frac{GM\rho_F d_F \cdot \pi R_F^2}{R_F^2} = \frac{GMm}{2R_F^2}.$$

The equation of the circular motion of the hemisphere

$$\sum m_i \omega^2 R_i = \frac{GMm}{2R_F^2}.$$

Here, for each m_i, you also need to count its contribution, and if you calculate the integral correctly

$$\sum m_i \omega^2 R_i = \frac{\pi}{4} m\omega^2 R.$$

$$\frac{\pi}{4} m\omega^2 R_F = \frac{GMm}{2R_F^2}, \quad \text{откуда} \quad R_F = \sqrt[3]{\frac{2GM}{\pi\omega^2}} = \sqrt[3]{\frac{GMT^2}{2\pi^3}}.$$

Calculations give 2.5 million km. (≈ 1/60 au).

But let's make a cylindrical approximation in this case and write

$$m\omega^2 R_F = \frac{GMm}{2R_F^2}, \quad \text{откуда} \quad R_F = \sqrt[3]{\frac{GM}{2\omega^2}} = \sqrt[3]{\frac{GMT^2}{8\pi^2}}.$$

Calculations give 2.3 million km. (≈ 1/65 au).

And if in the first case we considered not the projections of gravity, but the material point at a distance of R_F, we would get a well-known expression

$$m\omega^2 R_F = \frac{GMm}{R_F^2}, \quad \text{откуда} \quad R_F = \sqrt[3]{\frac{GM}{\omega^2}} = \sqrt[3]{\frac{GMT^2}{4\pi^2}}.$$

Calculations give ~3 million km. (≈ 1/50 au), this is too rough.

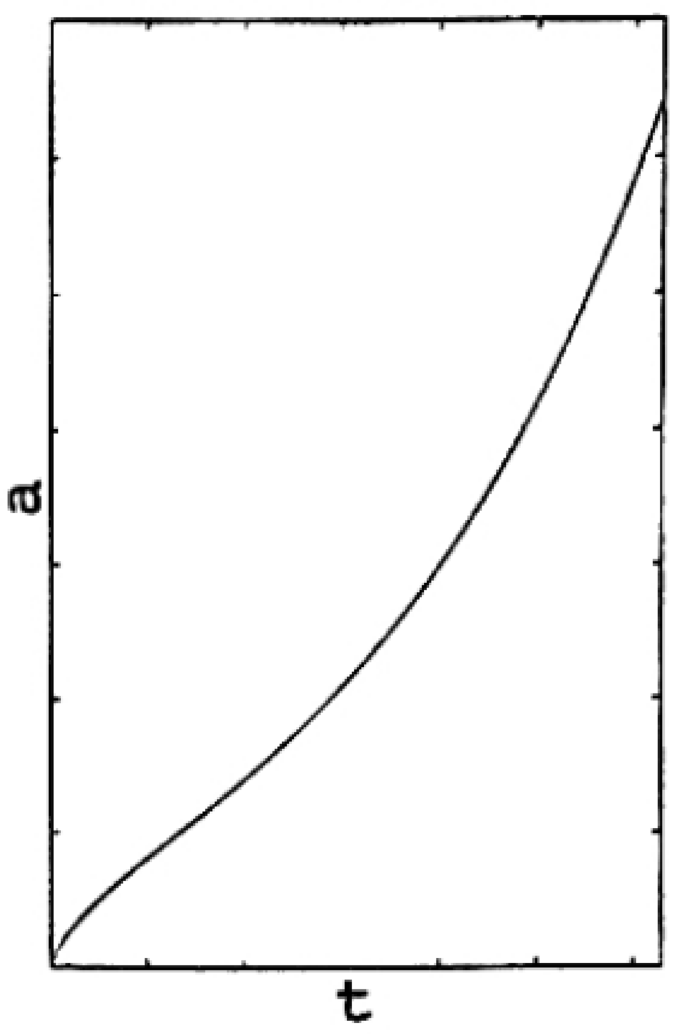
6.5-6. The orbits of Mercury, Venus and Mars (and the former orbit of the Earth, the substance of which was used in the construction of the sphere) will be completely inside the Dyson sphere. Due to the additional (compared to the current situation) energy flows circulating inside the Dyson sphere, climatic conditions will change on all planets inside the sphere. But if Mercury and Venus were already quite hot, then Mars will warm up more, because the temperature inside the sphere will not be lower than on its surface. Mars will probably become habitable.

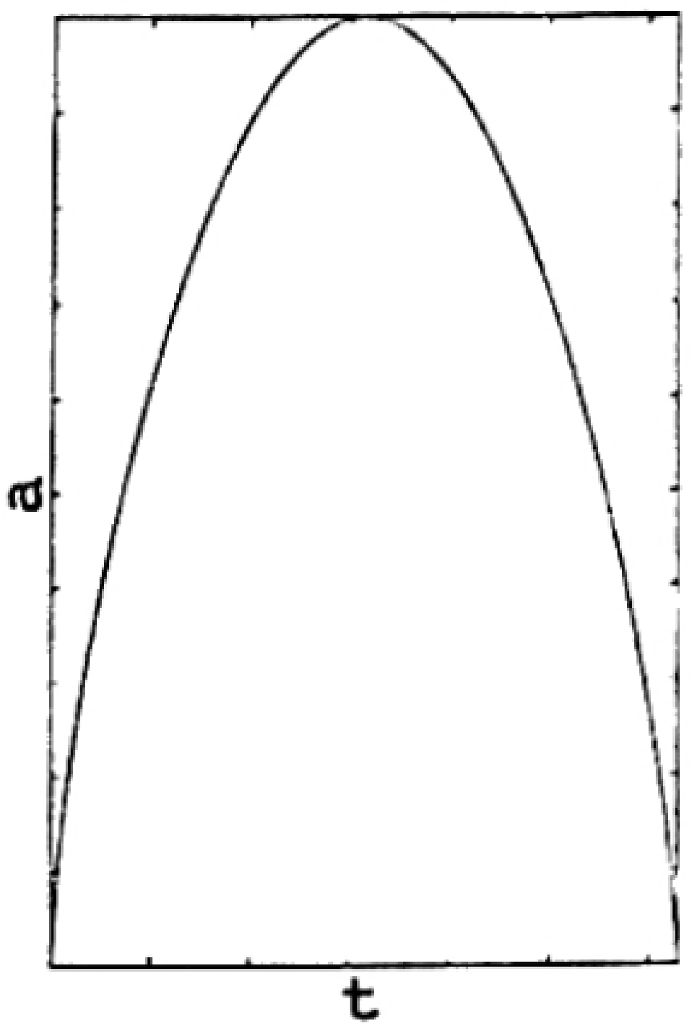
6.5. Mars.

6.6. Mercury, Venus.

7(βγ). Gravitational waves and cosmology.

7.1. Considering separately the cases of $\Lambda > 0$ and $\Lambda < 0$, the schematic diagram of the evolution of $a(t)$ is shown as follows. [1 point]





From the schematic diagram, it can be observed that when $\Lambda > 0$, the universe goes on (1) expansion; when $\Lambda < 0$, the universe goes on (3) expansion followed by contraction. [1 point]

7.2. For electromagnetic waves emitted by a radiation source, when considering two successive peaks, their emission times are t_e and $t_e + \Delta t_e$, while the times of receiving them by an observer are t_o and $t_o + \Delta t_o$, respectively. We have [0.5 points]

$$\frac{c\Delta t_e}{a(t_e)} = \frac{c\Delta t_o}{a(t_o)},$$

which means

$$\frac{\Delta t_e}{\Delta t_o} = \frac{a(t_e)}{a(t_o)}.$$

And the wavelength of light, λ, satisfies $\lambda = c\Delta t$. Therefore, [0.5 points]

$$\frac{\lambda_e}{\lambda_o} = \frac{a(t_e)}{a(t_o)}.$$

According to the definition of redshift, we have $\lambda_e/\lambda_o = 1 + z$, so the relationship between cosmological redshift and the scale factor is [1 point]

$$1 + z = \frac{a(t_e)}{a(t_o)}.$$

The luminosity L_e of the radiation source at time t_e is the energy radiated per unit time, i.e.

$$L_e = \frac{\Delta E_e}{\Delta t_e},$$

and this radiation is observed by the observer at time t_o. Considering cosmological redshift, the energy of the radiation becomes

$$\Delta E_o = \frac{\Delta E_e}{1 + z},$$

and the time interval for this radiation becomes [0.5 points]

$$\Delta t_o = \Delta t_e(1 + z).$$

For the observer, the brightness of the radiation is the energy received per unit area in unit observation time, i.e. [0.5 points]

$$F = \frac{\Delta E_o/\Delta t_o}{4\pi d^2},$$

where d is the physical distance between the radiation source and the observer. At observation time t_o, this physical distance is $d = a(t_o)r$, where r is the comoving distance between them. Combining the above equations, we find [1 point]

$$F = \frac{\Delta E_e/\Delta t_e}{4\pi[a(t_o)r]^2(1 + z)^2} \equiv \frac{L}{4\pi d_L^2},$$

so the luminosity distance at t_o is [0.5 points]

$$d_L = a(t_0)(1+z)r.$$

7.3. Considering cosmological redshift, the gravitational wave frequency seen by the observer at t_o becomes [0.5 points]

$$f_o = \frac{f_e}{1+z},$$

and the rate of change of frequency with time becomes

$$\dot{f}_o = \frac{\Delta f_o}{\Delta t_o} = \frac{\Delta f_e/(1+z)}{\Delta t_e(1+z)} = \frac{\dot{f}_e}{(1+z)^2}.$$

Therefore, we see that

$$\mathcal{M}_o \propto f_o^{-11/5}\dot{f}_o^{3/5} = f_e^{-11/5}\dot{f}_e^{3/5}(1+z),$$

i.e. [0.5 points]

$$\mathcal{M}_o = \mathcal{M}_e(1+z).$$

Furthermore, the distance of the gravitational wave source defined in the question satisfies

$$d_o \propto \mathcal{M}_o^{5/3} f_o^{2/3} = \mathcal{M}_e^{5/3} f_e^{2/3}(1+z),$$

which means [0.5 points]

$$d_o \propto 1+z.$$

This means that this distance is the luminosity distance.

Luminosity distance. [1 point]

Theoretical round. Basic criteria. For work of Jury

Note. Jury members should evaluate the student's solutions in essence, and not by looking on formal existence the mentioned sentences or formulae. The formal presence of the mentioned positions in the text is not necessary to give the respective points. Points should be done if the following steps de facto use these positions. (For example, except the final conclusion, all the points in solution of problem 2 may be in only one common formula.)

Note. Jury members should elaborate more detailed criteria, and also create criteria for other correct ways of the student's solutions.

Note. Everywhere "formula" means "correct formula", "calculation" means "**correct** calculation", "picture" means "**correct** picture", etc., and in all cases: based on **correct** facts and intermediate formulae or calculations.

Note. Everywhere final calculation means also using correct number of significant digits. Extra significant digits, especially in estimational problems, will lead to subtraction of the points.

Note. Solutions with absurd or almost absurd answers should be rated lower (maybe ignoring the scoring schemes below, and maybe down to zero).

Everywhere – excessive number of significant digits in the answer – minus 1-2 points.

1.	**Crab nebula**	pts
	The distance through a parallax	1
	Apparent (as we see) the physical size	1
	Rate of expansion, formulae, calculations	3
	Final answer $7\cdot10^{14}$ km, $5\cdot10^{6}$ au	1
	Correct number significant digits in answer	2
	Total	8

2.	**Space apparatus**	pts
	Speed of the apparatus	1.5
	Distance in visibility	0.5
	Limit altitudes	1
	Apparent atmospheric extinction	2
	Magnitude in ideal conditions	1
	Calculations and final answer	1
	Correct number significant digits in answer	1
	Total	8

3.	**Interstellar Travel**	pts
3.1.	Distance between the Sun and α Cen	0.5
	Flux of the Sun seen at α Cen	0.5
	Solid angle of the engine plume	0.5
	Flux from the ship	0.5
	General formulae and calculations	2
	Final answer	1.5
3.2.	Explanation and calculations	2
	Final conclusion “Yes”	0.5
	Total	8

4.	**Hybrid eclipse**	pts
	General understanding	1
	Calculation differences relative Earth center, its difference	1.5
	Cone angle	1.5
	Width of the cone	1.5
	Width of the path	1.5
	Final calculations and answer	1
	Total	8

5.	**Meteors**	pts
5.1.	Probability approach	3
	Coefficient of noticability	1.5
	Final formulae and calculations	1.5
5.2.	Picture	0.5
	Artistic character of picture	1.5
	Total	8

6.	**Dyson sphere**	pts
6.1.	Energy balance	0.5
	Formulae and calculation for ratio R_D	1
6.2.	Formulae and calculation for ratio d_D	1
6.3.	Formulae and calculation for ratio g_D	1
6.4.	Estimation formulae and calculations	2
	Accurate formulae and calculations	+1
6.5.	~~Finding diameter of white dwarf from the mass-radius dependence~~	1
6.6.	~~Finding diameter of white dwarf from the mass-radius dependence~~	0.5
	Total	8

7.	**Gravitational waves and cosmology**	pts
7.1.	→ See criteria in the solutions	2
7.2.	→ See criteria in the solutions	4
7.3.	→ See criteria in the solutions	2
	Total	8